Shell-model studies of Magnetohydrodynamic Turbulence in three dimensions

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Abstract—We present a systematic study of multiscaling properties of Magnetohydrodynamic (MHD) turbulence in three dimensions (3dMHD) for a shell model, developed by us. We define structure functions as appropriate for an MHD shell model and use them to uncover these scaling properties. We also discuss a crossover from fluid-like behaviour to MHD-like behaviour.

Keywords—MHD turbulence, Shell model, Multiscaling.

The extension of Kolmogorov’s work (K41) [1] on fluid turbulence to magnetohydrodynamic (MHD) turbulence yields [2] simple scaling for velocity \( \mathbf{v} \) and magnetic-field \( \mathbf{b} \) structure functions, for distances \( r \) in the inertial range, between the forcing scale \( L \) and the dissipation scale \( \eta_d \). Many studies have shown that there are multiscaling corrections to K41 scaling in fluid turbulence [3]. Solar-wind data [4], recent shell-model studies [5], [6] and direct numerical simulations (DNS) for MHD turbulence yield similar multiscaling. This paper is a short overview of the work that we have done in this area.

To establish notations, it is useful to begin with the equations of magnetohydrodynamics [2] in three dimensions (3dMHD) for the coupled evolution of the velocity field \( \mathbf{v} \) and the magnetic field \( \mathbf{B} \). The MHD are like the Navier-Stokes equation for an incompressible fluid, but modified by the inclusion of electromagnetic stresses. They are (in CGS units):

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi \rho} + \nabla^2 \mathbf{v} + \mathbf{f}_e,
\]

with \( \nu, p, \rho, \) and \( \mathbf{f}_e \) are the kinematic viscosity, pressure, density and external force respectively. Ampère’s law for a conducting fluid becomes

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \mu \nabla^2 \mathbf{B} + \mathbf{f}_m,
\]

with \( \mu \) and \( \mathbf{f}_m \) are the magnetic viscosity and external force respectively. \( \mu \) is inversely proportional to the conductivity of the medium. Of course, \( \nabla \cdot \mathbf{B} = 0 \) and, if the fluid is incompressible, then \( \nabla \cdot \mathbf{v} = 0 \). In the latter case the effective pressure \( p^* = p + B^2/(8\pi) \) can be eliminated as usual.

Studies of MHD turbulence often use the real-space structure functions \( S^a_p (r) \) of a fluid, or one of the Elsässer variables \( \mathbf{Z}^\pm = \mathbf{v} \pm \mathbf{b} \), and the angular brackets denote an average over the statistical steady state. \( S^a_p \approx r^{C_p^a} \) at high fluid and magnetic Reynolds numbers \( Re \) and \( Re_b \), respectively, and for the inertial range \( \eta_d \ll r \ll L \). By extending the K41 results for fluid turbulence it immediately follows from dimensional analysis that structure functions will have the form \( S^a_p \approx C_p^a (r) r^{C_p^a/3} \). Hence, at the level of K41 we get \( C_p^a = p/3 \). The result \( C_p^a = p/3 \) in fluid turbulence is believed to hold exactly and follows from the von Karman-Howarth relation [8]. The analogue of this has been derived for MHD turbulence in Ref.[18]. Hence it is reasonable to expect \( C_p^a = 1 \) in MHD turbulence. The K41 result \( C_p^a = 2/3 \) implies kinetic- and magnetic- energy spectra \( E^a(k) = C_\alpha (k) k^{-5/3} \), where \( k \) is the wavevector and \( \alpha \) the mean energy dissipation in the statistical steady state. This is the MHD analogue of the famous -5/3 law of fluid turbulence. Shell models [5], [6] and solar-wind data [4] have obtained multiscaling in MHD turbulence, i.e., \( C_p^a = p/3 - \delta C_p^a \), with \( \delta C_p^a > 0 \) and \( C_p^a \) nonlinear, monotonically increasing functions of \( p \). In shell-model studies it is convenient to use wavevector space (\( k \)-space) analogues of \( S_p^a (r) \) which we define below.

Direct Numerical Simulation of Eqs.(1) and (2) are difficult since, at large Reynolds numbers (\( Re \)) scale as \( R^{5/4} \) and computational time as \( R^3 \). These difficulties have led to the construction of simpler shell models for numerical studies of fluid turbulence. The Gledzer-Okhitani-Yamada (GOY) is the most well-known of them; numerical studies of this yield the same multiscaling exponents, within error bars, as those from the Navier-Stokes equation. Similar attempts to construct shell models for MHD turbulence have used the following guidelines:

1. Since we wish to concentrate on scaling and multiscaling properties of structure functions, and not, say, coherent structures, it should suffice to use a simplified model in which the shell velocity and magnetic fields, or equivalently, the Elsässer variables \( \mathbf{z}^\pm \) are complex, scalar quantities. The shells are labeled by scalar wavevectors \( k_n \) in a logarithmically discretised, one-dimensional \( k \)-space.

2. The shell-model equations should preserve the \( \mathbf{Z}^+ \leftrightarrow \mathbf{Z}^- \) symmetry of the 3dMHD equations and their conservation laws. In particular, we should conserve the shell-model analogues of total energy, cross helicity and magnetic helicity in the absence of any viscosity and external forces.

3. It should be invariant under the parity transformation that leaves the 3dMHD equations unchanged: The 3dMHD equations are invariant under \( \mathbf{x} \rightarrow -\mathbf{x}, \mathbf{v} \rightarrow -\mathbf{v}, \mathbf{B} \rightarrow -\mathbf{B} \). Hence, our shell-model equations should be invariant under \( k_n \rightarrow -k_n, b_n \rightarrow -b_n \).

4. The MHD shell model should reduce to a fluid shell model in the absence of any magnetic field. To the best of our knowledge this has been imposed only in the MHD shell model studies of Refs.[12], [14].

5. In the inviscid, unforced limit, stationary solutions of the equations should scale as \( v_n \sim k_n^{-1/3}, b_n \sim k_n^{-1/3} \). The MHD shell-model equations have the form

\[
\frac{dz^\pm}{dt} = i c_n^\pm - v_n k_n^2 z^\pm_n - v_n k_n^2 z^\mp_n + f_n^\pm.
\]

The earliest MHD shell model [13] used real, scalar Elsässer...
variables $z_{n^+}^\pm$, but most recent MHD shell models [5], [12], [14] use the complex, scalar Els"asser variables $u_n^\pm \equiv (v_n \pm b_n)$, and discrete wavevectors $k_n = k_0 \lambda^n$, for each shell $n$. Different shell models of MHD are distinguished by different forms for the nonlinear terms $c_n^\pm$. In particular, the model developed independently in Ref.[12], [14] is defined by the nonlinear terms

$$c_n^\pm = [a_1 k_n z_n^+ z_{n+1}^- + a_2 k_n z_n^- z_{n+1}^+ + a_3 k_n z_n^+ z_n^- + a_4 k_n z_{n+1}^- z_n^+] + a_5 k_n z_{n-1}^+ z_{n+1}^- + a_6 k_n z_{n+1}^- z_{n-1}^+$$

They ensure that $z_{n+1}^+ z_{n+2}^- \sim k_n^{-1/3}$ is a stationary solution of Eqs.(3) in the inviscid, unforced limit. In Ref.[12] there was a typographical error which we correct here - this leaves all the results in Ref.[12] unchanged.

We fix five of the parameters, $a_1 - a_6$, by demanding that our shell-model analogues of the total energy ($\equiv \sum_n |v_n|^2 + |b_n|^2/2$), the cross helicity ($\equiv 1/2 \sum_n (v_n b_n^* + v_n^* b_n)$), and the magnetic helicity ($\equiv \sum_n (-1)^n |b_n|^2 / k_n$) be conserved if $\nu = 0$ and $f = 0$; while enforcing the conservation of energy, we also demand [12] that the cancellation of terms occurs as in $3d$MHD. We obtain $A_1 = a_1 + a_2 = 1$, $A_2 = a_3 + a_4 = \varepsilon - 1$, $A_3 = a_3 + a_6 = -\varepsilon$, for arbitrary $\varepsilon$ and $a_1 - a_2 = -(a_5 - a_6)/\lambda^3 = (a_5 - a_4)/\lambda$, with (define $A_4 = a_1 - a_2$, $A_5 = a_3 - a_4$, $A_6 = a_5 - a_6$, $b_n = k_0 \lambda^n$).

In addition, we demand that our shell model must reduce to the GOY shell model for fluid turbulence in the absence of any magnetic field. All the above conditions finally yield $\varepsilon = 1 - 1/\lambda$ (as in the GOY shell model) and $\varepsilon = 1/\lambda$ which together yield $\varepsilon = 1/2$ and $\lambda = 2$. Thus, apart from the Reynolds numbers, our shell model has no adjustable parameters and $a_1 = 7/12$, $a_2 = 5/12$, $a_3 = -1/12$, $a_4 = -5/12$, $a_5 = -7/12$, $a_6 = 1/12$, and $\lambda = 2$.

We define the shell-model analogue of the structure functions as $S^a_n(k_n) = \langle |a(k_n)|^p \rangle$. As in the GOY shell model, the MHD shell-model static solutions exhibit three-cycles: we as-

$$A_1 k_n q_{n-1} q_{n+2} + A_2 k_n q_{n+2} - A_3 k_n q_{n-2} \quad (A_1 k_n s_{n+1} s_n + q_{n-1} q_{n+1} - A_2 k_n s_{n+1} s_n + A_3 k_n s_{n-1} s_n - A_4 k_n s_{n-1} s_n + A_5 k_n s_{n+1} s_{n+1} - A_6 k_n s_{n+1} s_{n+1} - A_7 k_n s_{n+1} s_{n+1} = 0)\quad (4)$$

By a numerical iteration of this set of difference equations (4) and (5) we find that, at the fixed points, $s_n = s_{n+1} = s^*$ for all $n$ where $s^*$ is real. The magnitude of $s^*$ is determined by the initial conditions. We thus set $s_n = s^*$ and look for fixed points of $q_{n-1} q_{n+1} = q^*$ of the remaining equation. It is easy to see that these are fixed points of the following one-dimensional complex ratio-map (for an analogous treatment for the GOY shell model see Ref.[16]): $q_n = q_n = k_n^{-1/3} g_1^4(n)$ and $q g_1 = (1/2) \rightarrow u_n \sim u_n^{K_{44}} = k_n^{-1/3} g_1^4(n)$. For the numerical scheme, we use to solve Eqs.(3) we refer the reader to Ref.[12], [19]. The parameters in our different runs are given in Table 1. We also compare some of our results by DNS of $3d$MHD equations.

![Fig. 1. Plots of log $S^a_n(k_n)$ versus shell number $n$ for $p = 1, 3, 5, 7, 9, \alpha = v, b$ (from top to bottom) for (i) $a = v$, (ii) $a = b$; (iii) A plot of log $S^a_n(k_n)$ versus log $k_n$ from run SH1.]

We study the scaling properties of $S^a_n(k_n)$ and $S^b_n(k_n)$ to find the analogue of the Karman-Howarth result for MHD turbulence. In Fig.1(iii) we show a log-log plot of $S^a_n(k_n)$ for $a = v, b$ versus $k_n$. We find that $\zeta^a_3 \simeq \zeta^b_3 \simeq 1$ within our error bars. Thus our results are in agreement with [18]. In the inertial
range \( k_0 \ll k \ll k_d \sim \eta_0^{-1} \). The structure functions exhibit the multiscaling behaviour characterised by the exponents \( \zeta_p^a \). However, as one approaches \( k_d \), viscosity starts becoming important and the structure functions lose this simple power-law behaviour. Several studies \cite{10} have been shown that, by using ESS, structure functions exhibit a power-law dependence on \( k \) in an extended inertial range. We begin with the following ESS ansatz: 
\[
\Sigma_p^a \approx A_p^\alpha (\Sigma_0^2)^{\alpha^2/\alpha^2}, \quad L^{-1} \approx k \ll k_d,
\]
where \( \alpha(k) \) is the Fourier transform of \( \alpha(r) \), \( A_p^\alpha \) are nonuniversal amplitudes. The universal exponents \( \zeta_p^a \) characterise the asymptotic behaviours of the structure functions in the inertial range. In Fig. (2) we show plots of \( \zeta_p^a / \zeta_p^b \) versus \( p \) and their comparison with the inertial range exponents given by the She-Leveque formula which is a convenient parametrization for the multiscaling exponents for homogeneous, isotropic fluid turbulence. The formula is
\[
\zeta_p^{SL} = \frac{p}{3} + 2 \left[ 1 - \left( \frac{2}{3} \right)^{p/3} \right].
\]

We follow the method used by Dhar et al \cite{15} to investigate the non-Gaussian behaviour of shell-model structure functions. Here we restrict ourselves to \( k \)-space quantities for our MHD shell model and define
\[
\Gamma_p^a (n) = \frac{(2p - 1)! (\Sigma_0^2 (k_n))^p}{2 \Sigma_2^a (k_n)}, \quad p = 2, 3, 4, \ldots; \quad a = v, b.
\]

If the Probability distributions (PDF) of \( |v_n| \) and \( |b_n| \) were Gaussian, \( \Gamma_p^a (n) \) would have been \( 1/2 \) for all \( p \). In Fig. (3) we plot \( \log \Gamma_p^a (n), a = v, b \) as a function of \( n \), the shell number, for the run SH1. One can clearly observe the deviation of \( \Gamma_p^a (n) \) from the Gaussian value of \( 0.5 \) in the inertial range and its rapid decay in the dissipation range (a manifestation of the strong non-Gaussian tail of the PDFs in the dissipation range).

The general conclusions that we can derive from these plots is that the PDFs \( P(|a_n|) \) crosses over from being close to Gaussian at small \( k_n \) to one that has a tail that decays more and more slowly with increasing \( k_n \). Figures (3) also illustrate that, with increasing Reynolds numbers, the extent of the inertial range increases and that the deviations from Gaussian distributions are far more in the dissipation range than in the inertial range.

We have already pointed out that our shell model reduces to the GOY shell model in the absence of any magnetic field. We

Fig. 2. Inertial-range exponents versus \( p \) from typical 3d MHD and shell-model runs and their comparison with the SL formula.

Fig. 3. \( \log \Gamma_p^a (n), a = v, b \) as functions of shell number \( n \) from runs SH1, SH2 for \( 2p = 4, 6, 8, 10 \).
have used this property of our model to study a crossover from the fluid- to MHD-turbulence behaviour. We chose, at $t = 0$, all $b_k = 0$ and evolve the equation for $v_3$ till we reach the ordinary fluid statistical steady state exhibiting fluid multiscaling. Then we introduce a very small magnetic field in our MHD shell model. We observe that the magnetic field starts growing till the MHD steady state is reached. We measure the fluid multiscaling exponents during this transient behaviour and observe that the fluid-turbulence exponents eventually cross over to their MHD values [see Fig.(4)]. Thus, in a renormalisation-group calculation $R_{E(k)}$ should appear as a relevant operator that takes the system from the fluid-turbulence fixed point to the MHD-turbulence fixed point.

Thus, in conclusion, we elucidated the properties of the statistical steady-state of an MHD shell model, characterized by a set of multiscaling exponents. In particular we examined the structure and the guiding principles of such a model and how to fix the parameters in it by imposing various conditions on it. We then discussed the scaling of the structure functions in the inertial range and compare our results with representative data from the numerical solutions of the 3dMHD equations. We also illustrated the non-Gaussian nature of the PDF of the velocity and magnetic fields in the MHD shell models.

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REFERENCES

[5] D. Biskamp, Phys. Rev. E, 50, Cascade models for magnetohydrodynamic turbulence, D. Biskamp,2702 (1994) also finds $\bar{Q} > \bar{Q}_e$, for $p = 2$, in a shell model.

TABLE I

The viscosities and hyperviscosities $\nu$, $\mu_b$, $\mu_H$, and $\mu_{SH}$, the Taylor-microscale Reynolds numbers $Re_\lambda$ and $Re_k$, the box-size eddy-turnover times $\tau_\lambda$ and $\tau_k$, the averaging time $\tau_A$, the time over which transients are allowed to decay, $\tau_s$ and $k_d$ (dissipation-scale wavenumber) for our 3dMHD runs ($k_{max} = 32$ for MHD1 and $k_{max} = 40$ for MHD2) and shell-model runs SH1-4 ($k_{max} = 2^{30} k_0$). The step size($\delta t$) is 0.02 for MHD1-4, $2 \times 10^{-5}$ for SH1-2, and $10^{-4}$ for SH3-4. Note that $\tau_s \approx 8 \tau_A$ the integral time for our MHD runs.

<table>
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<th>Run</th>
<th>$\nu$</th>
<th>$\nu_H$</th>
<th>$\mu$</th>
<th>$\mu_H$</th>
<th>$Re_\lambda$</th>
<th>$Re_k$</th>
<th>$\tau_\lambda/\delta t$</th>
<th>$\tau_k/\delta t$</th>
<th>$\tau_s/\tau_A$</th>
<th>$k_{max}/k_d$</th>
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<td>$8 \times 10^{-4}$</td>
<td>$7 \times 10^{-6}$</td>
<td>$10^{-6}$</td>
<td>$8 \times 10^{-6}$</td>
<td>$\approx 24.8$</td>
<td>$\approx 14.3$</td>
<td>$\approx 8.8 \times 10^{3}$</td>
<td>$\approx 6 \times 10^2$</td>
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<tr>
<td>MHD2</td>
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<td>$9 \times 10^{-6}$</td>
<td>$8 \times 10^{-4}$</td>
<td>$9 \times 10^{-6}$</td>
<td>$\approx 26$</td>
<td>$\approx 39.6$</td>
<td>$\approx 7.9 \times 10^2$</td>
<td>$\approx 4.8 \times 10^2$</td>
<td>$\approx 1$</td>
<td>$\approx 2.2$</td>
</tr>
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<td>$10^{-6}$</td>
<td>0</td>
<td>$10^{-8}$</td>
<td>0</td>
<td>$\approx 4.6 \times 10^4$</td>
<td>$\approx 7.8 \times 10^3$</td>
<td>$\approx 7 \times 10^7$</td>
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<td>$\approx 450$</td>
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<td>0</td>
<td>$10^{-8}$</td>
<td>0</td>
<td>$\approx 4.3 \times 10^5$</td>
<td>$\approx 6.5 \times 10^7$</td>
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<td>$\approx 6 \times 10^6$</td>
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<td>0</td>
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<td>$\approx 2 \times 10^6$</td>
<td>$\approx 500$</td>
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<td>$\approx 500$</td>
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<tr>
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<td>$10^{-6}$</td>
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Fig. 4. A plot of $\bar{Q}_e^2/\bar{Q}_e^2$ versus time $t$ exhibiting its crossover from its fluid-turbulence value to MHD-turbulence value. At $t = 0$ it has the fluid value whereas at by $t = 50$, it has attained its MHD value. The large size of the error bars arises because we cannot average for long enough since the crossover is very rapid. In this plot time is measured in units of $0.001 \tau_A$. 


