Non–Linear Dynamics of Railway Wheel Set

Y. Nath, K. Athre, K. Jayadev

Abstract- The Governing equations of motion of railway wheel set moving on a straight track are derived using Lagrangian approach. The resulting nonlinear equations of motion of railway wheelset are solved using Fourth order Runge-Kutta method. Johnson and Vermeulen theory for creepage is employed. A general-purpose program is developed for the analysis. The results are presented in the form of time series diagram, Phase plots, Poincare plots, Bifurcation diagrams and Lyapunov exponent diagrams. The influence of yaw stiffness on the response of wheelset is studied. Both symmetric and asymmetric oscillations and chaotic motions are observed. The broadband chaos is developed at various velocity levels due to flange contact. Periodic windows were observed in the bifurcation diagram.

Keywords- Chaotic oscillation, Numerical Simulation, Railway Wheel set.

I. INTRODUCTION

Wheelset is the basic component of the railway vehicle systems. It is responsible for safe and comfortable transportation. The present paper is to explore the rich dynamics associated with it. The chaotic motion of a railway wheelset with wheels having cylindrical tyres, moving at a constant forward velocity is investigated by Meziard and De Pater [1]. They found two principle kind of motions. One in which flange contact takes place at one rail and one in which flange contact at both rails occur. Using Johnson and Vermeulen contact theory Knudsen, Rose and True [2] analyzed a model of suspended rolling railway wheelset. They investigated the effect of speed and suspension on the dynamics of wheelset. They presented the results in the form of time series and bifurcation diagrams. Silvsgaard and True [3] presented the results of the dynamics of a rolling wheelset that was started by Knudsen et al [2]. They found that the wheelset oscillates chaotically and identified the positions of periodic widows in the selected region accurately. Hitherto the effect of yaw stiffness is not considered.

In the present study nonlinear relation between creepage and creep, force based on Johnson and Vermeulen [4] contact theory has been employed for the estimation of contact forces between rail and wheel. Equations of motion are derived using Lagrange’s principle. These equations of motion of wheelset are solved using fourth order Runge-Kutta method. Results are presented in the form of time history, Poincare maps and bifurcation diagrams. Both symmetric and asymmetric oscillations and chaotic motion are observed. The influence of yaw stiffness and axial velocity on the dynamic response of wheelset is studied. Broadband chaotic motion is developed at various velocity levels. Intermittency and periodic doubling are observed in bifurcations with the increase or decrease in velocity.

II. MATHEMATICAL FORMULATION

The wheel set has two wheels rigidly connected with an axle. The wheels are the frustums of a cone with slope, λ. The wheels roll on rail surfaces, which are assumed to be an arc of a circle with radius r. The restoring force from the flanges on the wheels is approximated by a strong linear spring with a dead band and no damping. The lateral motion is restricted by linear springs with no damping. The wheels and the axle are assumed to be rigid bodies. Fig.1. shows the geometry of the wheelset. The wheelset is assumed to have two degrees of freedom. One is lateral motion, ‘Y’ and another is yaw motion, ‘Ψ’. The two equations of motion of wheelset are coupled through the non-linear creep-force terms in the equations. The resultant of the normal forces exerted between the wheels and the rails generally has a lateral component. It vanishes in this model, because the wheel profile is assumed to be conical. Friction is only included in the contact forces between the wheels and the rails. The nonlinear relation between creepage and creep force is taken according to Johnson and Vermeulen contact theory [4].

Kinetic energy of the system is

\[
K.E. = \frac{1}{2} \left( \frac{d\psi}{dt} \right)^2 + \frac{1}{2} m \left( \frac{dY}{dt} \right)^2
\]  (1)

Potential energy of the system is

\[
P.E. = 2 \left( \frac{1}{2} k_1 Y^2 \right) + 2 \left( \frac{1}{2} k_2 (d\psi)^2 \right)
\]  (2)

The Lagrange’s equation is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1, 2
\]  (3)

Where, \( L = K.E - P.E \)

Generalized forces are

\[
Q_1 = (-F_x + 2F_y)
\]

\[
Q_2 = -2aF_x
\]  (4)
Using above relations the equations of motion becomes

\[
m\frac{d^2 Y}{dt^2} + 2kY + 2F_v + F_r(Y) = 0
\]

\[
F_T\left(\frac{d^2 Y}{dt^2}\right) + 2k\frac{d^2 \Psi}{dt^2} + 2a(\Phi) = 0
\]

(6)

Where contact forces \( (F_v, F_r) \), force due to flange contact \( (F_T) \) and creepages are defined as

\[
F_x = \left(\frac{\xi_x}{\psi}\right) F_R / \xi_R
\]

\[
F_y = \left(\frac{\xi_y}{\psi}\right) F_R / \xi_R
\]

(7)

(8)

\[
F_T(Y) = \begin{cases}
  k_\phi (Y - \delta), & \delta < Y, \\
  0, & -\delta \leq Y \leq \delta, \\
  k_\phi (Y + \delta), & Y < -\delta,
\end{cases}
\]

(9)

\[
F_R = \mu N \left[ u - \frac{1}{3}u^2 + \frac{1}{27}u^3, \quad u < 3, \quad \right.
\]

\[
1, \quad \left. u \geq 3, \right]
\]

(10)

Where,

\[
\xi_y = \frac{Y}{v} - \psi
\]

(11)

\[
\xi_x = \frac{\lambda Y}{v^2} + \frac{a \psi}{v}
\]

(12)

\[
\xi_R = \sqrt{\left(\frac{\xi_x}{\Phi}\right)^2 + \left(\frac{\xi_y}{\Psi}\right)^2}
\]

(13)

Equations (5) and (6) are the governing equations of motion of wheel set for the lateral and yaw displacement. The list of constants used in the above equations is given in Table 1.

III. METHOD OF SOLUTION

The change of variables is made as given under,

\[
Y_1 = Y, \quad Y_2 = \dot{Y}, \quad Y_3 = \Psi, \quad Y_4 = \dot{Y}
\]

to obtain the following fourth order autonomous dynamical system.

\[
\ddot{Y}_1 = Y_2,
\]

(14)

\[
\ddot{Y}_2 = -2(\alpha/m)Y_1 - (2/m)F_Y - (1/m)F_T(Y_1)
\]

(15)

\[
\ddot{Y}_3 = Y_4,
\]

(16)

\[
\ddot{Y}_4 = -2(\beta_2l^2/I)Y_3 - 2(\alpha/l)F_X
\]

(17)

These autonomous equations are solved in time domain using fourth order Runge-Kutta method. A general-purpose software/program is developed in ‘C++’, for the analysis. Various plots are rendered using ‘Matlab’.

IV. RESULTS AND DISCUSSIONS

Convergence study revealed that \( \Delta t = 0.001 \) sec yield quite accurate results. Fig 2 shows the lateral response for \( V = 50 \) m/s for different values of yaw stiffness \( k_2 \). It can be noted that effect of \( k_2 \) on the response is insignificant and mushrooming of local maxima is not affected. It has been found that local maxima are generated with increasing forward velocity.

Fig 3 shows the phase plot for \( V = 50 \) m/s and \( k_2 = 180 \) N/m. Two attractors are quite distinct.
The Poincare section for $k_2=180 \text{ N/m}$ and $V=50 \text{ m/s}$ are depicted in Fig4. There are at least two distinct attractors around which clustering of points exist.

![Fig4. Poincare map of Wheel set at $k_2=180 \text{ N/m}$ and $V=50 \text{ m/s}$](image)

The Poincare section for $k_2=1800 \text{ N/m}$ and $V=50 \text{ m/s}$ are depicted in Fig5. Here also there are at least two distinct attractors around which clustering of points exist.

![Fig5. Poincare map of Wheel set at $k_2=1800 \text{ N/m}$ and $V=50 \text{ m/s}$](image)

Lyapunov exponent of the system is shown in Fig7. The positiveness of Lyapunov exponent is an indication of chaotic motion.

![Fig7. Lyapunov exponents at $k_2=1800 \text{ N/m}$ and $V=50 \text{ m/s}$](image)

**V. CONCLUSIONS**

Equations of motion of wheel set are solved using the fourth order Runge-Kutta method. The results presented are helpful to understand the complicated but important behavior of Wheelset and its dependence on axial velocity and yaw stiffness. After supercritical Hopf bifurcation, the first chaotic speed interval appeared. With growing speed the chaos alternates with periodic symmetric or asymmetric solutions. The dominance of asymmetry is noticed in the periodic behavior. The asymmetric solutions may cause lopside wear of the wheel set. In many cases chaotic oscillations develop as flange contact is obtained. Positive Lyapunov exponent is a clear-cut indication of chaotic motion. Broadband chaos started from a velocity of 26.2 m/s at zero yaw stiffness and many periodic windows appeared with the increasing speed.

Initial point of broadband chaos increased and high asymmetry in periodic oscillations is noticed with the increase in yaw stiffness.
# TABLE 1

**LIST OF PARAMETERS [4]**

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>1022 kg</td>
<td>Mass of wheel axle</td>
</tr>
<tr>
<td>I</td>
<td>678 kg m²</td>
<td>Moment of inertia</td>
</tr>
<tr>
<td>A</td>
<td>0.716 m</td>
<td>Half of the track gauge</td>
</tr>
<tr>
<td>(d_1)</td>
<td>0.620 m</td>
<td>Distance from center of gravity to (k_2)</td>
</tr>
<tr>
<td>G</td>
<td>808 MN m⁻²</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>(a_e)</td>
<td>6.578 mm</td>
<td>Major semi axis of contact ellipse</td>
</tr>
<tr>
<td>(b_e)</td>
<td>3.934 mm</td>
<td>Minor semi axis of contact ellipse</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>9.1 mm</td>
<td>Dead band</td>
</tr>
<tr>
<td>(k_0)</td>
<td>14.60 MN m⁻¹</td>
<td>Spring constant (flange)</td>
</tr>
<tr>
<td>(k_1)</td>
<td>18.23 kN m⁻¹</td>
<td>Spring constant (lateral)</td>
</tr>
<tr>
<td>(\Psi)</td>
<td>0.54219</td>
<td>Constant</td>
</tr>
<tr>
<td>(\Phi)</td>
<td>0.60252</td>
<td>Constant</td>
</tr>
<tr>
<td>(r_0)</td>
<td>0.4572 m</td>
<td>Centred wheel rolling radius</td>
</tr>
<tr>
<td>(\Lambda)</td>
<td>0.05</td>
<td>Slope of conical wheel</td>
</tr>
<tr>
<td>(\mu)</td>
<td>0.15</td>
<td>Coefficient of friction</td>
</tr>
<tr>
<td>(\mu_N)</td>
<td>10 kN</td>
<td>N is the vertical force between wheel and rail</td>
</tr>
</tbody>
</table>

## VI. NOMENCLATURE

- \(Y, \dot{Y}\) Lateral displacement and Velocity
- \(\Psi\) Yaw angle
- \(\dot{\Psi}\) Yaw angular velocity
- \(\xi_Y\) Lateral creepage
- \(\xi_X\) Longitudinal creepage
- \(\xi_R\) Resultant creepage
- \(F_x\) Longitudinal creep force
- \(F_Y\) Lateral creep force
- \(F_R\) Resultant creep force
- \(\Psi, \Phi\) Coefficients of Johnson’s formula
- \(F_f(y)\) Flange contact force
- \(k_1, k_2\) Lateral, Yaw Spring constants

## ACKNOWLEDGEMENTS

The Research is partially supported by the Council of Scientific and Industrial Research, New Delhi, Grant No. 22(0315)/00/EMR-II. The support is gratefully acknowledged.

## REFERENCES