Enhancing Chaotic Phase Synchronization by forcing External Chaos

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Abstract- Phase synchronization is observed in coupled chaotic systems in the weaker coupling limit. Noise plays a constructive role for weaker coupling outside the phase synchronization regime. In this paper, the authors tried to understand whether chaos can also play constructive role since chaos has a broadband spectrum as similar to noise. Numerical experiments show that phase synchronization in coupled Rössler systems with different characteristic time can be enhanced by adding chaos from external sources like Chua’s oscillator and Lorenz systems.

Keywords- Phase synchronization, Rössler oscillators, Chua’s oscillator, Lorenz system, Noise, Chaos.

I. INTRODUCTION

Phase synchronization (PS) has been observed [1-5] in many weakly interacting chaotic systems. It has been an important field of study since the identification of such phase coherent behavior in cardio-respiratory interaction [6] and in neuronal system [7,8]. PS in coupled chaotic oscillators has been observed above critical coupling, εc<ε<εs, when the phases of the interacting systems are locked to each other although the amplitudes are uncorrelated. The phases φ1,2 of the coupled oscillators satisfy the relation |nφ1-mφ2|<constant for n:m phase locking (n, m are integers). In the simplest case of 1:1 phase locking (m=n=1), |φ1-φ2|<constant and thus remains bounded in time. Regular phase slips occur outside PS region (εc<ε<εs). Noise is otherwise destructive to PS in periodic oscillators by inducing phase slips but, recently, it has been reported [9-10] as to play constructive role too in chaotic oscillators similar to stochastic resonance [11]. The phases of interacting systems outside the PS region remain locked with fewer phase slips for optimal noise intensity. Noise and chaos has similarities in behavior in terms of long term unpredictability and broadband spectrum. The question thus arises amongst neurocientists [12] and engineers [13] whether chaos works better than noise? In other applications, the performance of some systems and measurement improves with suitably added noise [14]. The author intends to address the problem whether chaos can play as similar role as noise in enhancing PS. If it comes true, chaos can be used instead of noise for improving system performance with a better control over deterministic chaos than noise.

In this paper, the authors addressed the problem of enhancing PS in coupled chaotic oscillators by forcing chaotic signal of an optimal intensity.

Two Rössler oscillators are weakly coupled outside PS regime and chaotic signals are forced to each of them from two independent Chua’s oscillators. Numerical simulations show that PS is enhanced depending upon the strength and characteristic time scale of external chaos.

II. COUPLED RÖSSLER FORCED BY CHAOTIC SIGNAL

Two Rössler oscillators with mismatched parameters are coupled bidirectionally. The parameter mismatch introduces a frequency disorder between the two oscillators. The governing equations of the oscillators are given by

\[
\frac{dx_{1,2}}{dt} = -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon (x_{2,1} - x_{1,2})
\]

\[
\frac{dy_{1,2}}{dt} = \omega_{1,2} x_{1,2} + 0.15 y_{1,2} + \sigma_{1,2} (x_{1,2})
\]

\[
\frac{dz_{1,2}}{dt} = 0.4 + (x_{1,2} - 8.5) z_{1,2}
\]

where \(\omega_1=0.97\) and \(\omega_2=0.99\), and \((x_{1,2})\) is the forcing signals from two independent chaotic sources as Chua’s oscillators. \(\varepsilon\) is the strength of coupling and \(\sigma_{1,2}\) decides the strength of forcing. \(\Delta\omega = \omega_1 - \omega_2\) is the frequency mismatch between two Rössler oscillators. The dynamics of Chua’s oscillators is governed by

\[
\frac{d(x_c)_{1,2}}{dt} = \tau_{1,2} [\alpha_{1,2} (x_{c,1,2}) - \alpha_{1,2} (x_{c,1,2}) + \alpha_{1,2} f(x_{c,1,2})]
\]

\[
\frac{d(y_c)_{1,2}}{dt} = \tau_{1,2} [\beta_{1,2} (y_{c,1,2}) - (y_{c,1,2}) + (z_{c,1,2})]
\]

\[
\frac{d(z_c)_{1,2}}{dt} = \tau_{1,2} [-\beta_{1,2} (x_{c,1,2}) - \gamma_{1,2} (z_{c,1,2})]
\]

and

\[
f(x_c) = bx_c + 0.5(a - b) (|x_c + 1| - |x_c - 1|)
\]

The parameters \((a, b)\) decide the slopes of the piecewise linear function \(f(x_c)\) of each Chua’s oscillator. A different set of parameters \((\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2})\) is chosen for each Chua’s oscillator. By varying \(\tau_{1,2}\) the characteristic time scale of the Chua’s oscillators could be adjusted easily. The phases of chaotic oscillators can be measured by different methods [1-3], but it is estimated here by using return time on a Poincaré surface. The instantaneous phase of a scalar signal is defined [15] by

\[
\varphi(t) = 2\pi m + 2\pi \frac{t - t_n}{t_{n+1} - t_n}, \quad t_n \leq t < t_{n+1}
\]

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where \( t_n \) is the time of the \( n \)-th crossing on the Poincaré surface. The mean frequency of chaotic motion is then determined by

\[
\Omega = \lim_{N \to \infty} \frac{2\pi N}{N(t)}
\]  

where \( t(N) \) denotes the total time required to complete \( N \)-turns on the Poincaré surface. For weaker coupling \((\varepsilon<\Omega_{PS})\) between the Rössler oscillators outside the PS region, the phase difference \( \Delta \phi=|\phi_1-\phi_2| \) jumps regularly while it remains bounded with fewer phase slips in the PS regime \((\varepsilon>\Omega_{PS})\). It is reported [9] that coupled oscillators outside the PS region show larger epoch time between phase slips in presence of either independent or common noise thereby indicating enhancement of PS. In this work, the phase and mean frequency of the coupled oscillator are determined for varying intensity of forcing chaos to realize the effect of external chaos on PS.

### III. NUMERICAL RESULTS

The attractors of coherent chaos of two uncoupled Rössler oscillators in absence of external forcing are shown in Fig.1.

![Fig.1 Projection of uncoupled (\( \varepsilon=0, \sigma_1, \sigma_2=0 \)) Rössler attractor on (a) \( x_1-y_1 \) plane for oscillator-1 and (b) \( x_1-y_2 \) plane for oscillator-2 (also without chaos)](image)

![Fig.2 Attractors of two Chua’s oscillators on x-y plane (upper row), and Fourier spectra are given below (lower row) their corresponding phase plane plots](image)

The chaotic attractors of Chua’s oscillators and their corresponding Fourier spectra are shown in Fig.2. The parameters for Chua’s oscillators are selected as

\( (a=11.72138, \, \beta_1=16.16836, \, \gamma_1=0.4083164, \, \tau_1=1/1.7) \)

\( (a=13.62138, \, \beta_2=16.16836, \, \gamma_2=0.4083164, \, \tau_2=1) \) and \( (a=-1.433087, \, b=-0.7725338) \) so that it can generate double scroll chaotic oscillations. The double scroll oscillations of Chua’s oscillators have zero mean and delta correlated similar to noise behavior.

The natural frequency of Chua’s oscillators are given as 1.30849 and 2.8417.

![Fig.3(a) Mean frequency difference (\( \Delta \Omega_{i,2} \)) with coupling (\( \varepsilon \)): solid line for no chaos forcing (\( \sigma_1=0, \sigma_2=0 \)) and dashed line for forcing chaos (\( \sigma_1=0.05, \sigma_2=0.12 \)) (b) corresponding mean frequency (\( \Omega_i \)) of each Rossler system (solid line for no chaos, dashed line for forcing chaos)](image)

The Rössler oscillators are coupled in the weaker coupling limit outside PS regime (\( \varepsilon=0.023 \)). In absence of external chaos, the mean frequency difference of the coupled oscillators is close to zero (\( \Delta \Omega=0 \)) for coupling strength \( \varepsilon=0.0245 \) indicating onset of PS as shown in Fig.3(a). But \( \Delta \Omega=0 \) for much weaker coupling \( \varepsilon=0.024<\Omega_{PS}=0.0245 \) when the chaotic forcing with optimal strength is applied to both Rössler oscillators. The individual frequencies \((\Omega_i, i=1,2)\) of the coupled oscillators under chaos forcing are shown in Fig.3(b). It indicates enhancement of PS since frequencies of individual oscillators converge at a coupling lower than the critical coupling \( (\varepsilon<\Omega_{PS}, \text{critical coupling without forcing}) \). The phase difference (expressed in integers, \( \Delta \phi/2\pi \)) is shown in Fig.4 for different strength of chaos decided by \( \sigma_1, \sigma_2 \). It shows regular phase slips for no chaos forcing (D) but fewer phase slips with larger epoch may be observed with increasing strength of external chaos. It may be noted that the coupling strength of Rössler oscillators are chosen outside PS region (\( \varepsilon=0.023<\Omega_{PS} \)) before forcing. \( \Delta \Omega \) is quite large for this coupling as shown solid line in Fig.3(a). The phase differences in trace B \((\sigma_1=0.05, \sigma_2=0.11)\) and trace C \((\sigma_1=0.05, \sigma_2=0.12)\) in Fig.4 show much larger epoch between phase slips. This indicates enhancement of PS outside this regime \((\varepsilon=0.023<\Omega_{PS}) \) when chaos is forced. For even stronger external chaos in trace A \((\sigma_1=0.05, \sigma_2=0.15)\), phase slips with shorter epochs started appearing. PS is gradually destroyed with higher forcing strength. In fact, an optimal forcing strength is to be applied for enhancing PS.
The mean frequency difference ($\Delta \Omega$) is plotted with chaos intensity ($\sigma_2$) in Fig.5, which shows a minimum at optimal chaos intensity for $\sigma_2=0.12$.

The PS regime is thus clearly enhanced by forcing independent chaotic signal to each Rössler oscillator. Instead of using two Chua’s oscillators, one Chua’s oscillator and another Lorenz system is also be used for the purpose of forcing independent chaotic signal to the coupled Rössler, when similar PS enhancement has been observed.

IV. CONCLUSION

Added noise of optimal intensity can enhance PS in coupled chaotic oscillators, which has recently been observed by others. Since chaos has noise-like random behavior with broadband spectra, the authors tried to understand whether chaos could replace noise in order to achieve enhancement of PS. The results are highly encouraging, although the mechanism of such phenomena is yet to be explained. The effect of noise on PS is more or less understood by this time. Further work is necessary to establish the constructive role of chaos in enhancing PS for future applications. Chaos is more useful in practical applications since it is easier to control than noise.

ACKNOWLEDGEMENTS

This work is partly supported by BRNS/DAE under Grant no.2000/34/13/BRNS. The authors wish to thank C.S.Zhou for valuable suggestions and help in computing.

REFERENCES
