Occurrence of multiple attractor bifurcations in piecewise smooth systems

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Abstract—Multiple attractor bifurcations that are known to occur in piecewise smooth maps are a matter of serious concern in some physical and engineering systems. However, the conditions of occurrence of multiple attractor bifurcations have not work out to date. In this paper we report the results of our investigations in that direction.

Keywords—Piecewise smooth maps, Border collision bifurcation, Multiple attractor bifurcations.

Multiple attractor bifurcations are said to occur when multiple coexisting attractors are simultaneously created at a bifurcation point [1]. It has been shown earlier that in some cases border collision bifurcations (BCB) may lead to multiple attractor bifurcations [2]. BCBs occur when a fixed point of a piecewise smooth map collides with the borderline between two smooth regions [3]. Such bifurcations are very common in hybrid dynamical systems (where continuous-time evolution is punctuated by discrete events that alter the dynamical equations) and has been well studied in the context of power electronic circuits [4], [2].

If a parameter is smoothly varied resulting in a multiple attractor bifurcation, immediately after the bifurcation the separation between the coexisting attractors can be arbitrarily small. If there is noise present in the system, this results in a new and fundamental source of unpredictability: no matter how small the noise level is, it is impossible to predict which attractor the orbit will follow [1].

In practical systems, therefore, it is necessary to avoid the multiple attractor bifurcations. Development of control methodologies to avoid such bifurcations demand a theoretical understanding of the conditions under which multiple attractor bifurcations occur.

Detailed classification of border collision bifurcations in 1D and 2D systems have been developed in [2], [5], and some important results on the existence of fixed points and period-2 orbits in general n-dimensional maps have been reported in [6]. It has been shown that multiple attractor bifurcations cannot occur in one-dimensional piecewise smooth and continuous maps, but can occur in two-dimensional systems. But the past work on border collision bifurcations do not address the issue of occurrence of high periodic orbits in piecewise smooth systems. In this paper we critically examine the conditions of occurrence of stable coexisting orbits in the neighborhood of a border collision in 2D piecewise smooth maps.

It has been shown earlier that a subclass of hybrid dynamical systems can be represented in discrete-time by piecewise smooth maps, given by equations of the form

\[ f(x, y, \mu) = \begin{cases} 
g(x, y, \mu), & (x, y) \in R_A \\
h(x, y, \mu), & (x, y) \in R_B \end{cases} \]

where \( \mu \) is the bifurcation parameter and \( R_A \) and \( R_B \) are smooth regions in the state space, divided by a borderline. The map has the following properties:
1. The functions \( g \) and \( h \) are smooth (everywhere differentiable),
2. The function \( f \) is continuous across the borderline,
3. The elements of the Jacobian matrix of \( f \) change discretely across the borderline,
4. The Jacobian elements are finite.

Maps of the above properties have application in many physical and engineering systems, and in the present paper we restrict our attention to such maps.

Since the nature of border collision bifurcations depends on the local character of the map in the neighborhood of the fixed point, it suffices to look at the piecewise linear approximation at the two sides of the border. It has been shown [7], [2] that in the neighborhood of a border crossing fixed point the piecewise smooth map can be approximated by the piecewise linear normal form

\[
\begin{pmatrix}
x_{k+1} \\
y_{k+1}
\end{pmatrix} = \begin{cases} 
\begin{pmatrix} \tau_L - \delta_L \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \leq 0 \\
\begin{pmatrix} \tau_R - \delta_R \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \geq 0
\end{cases}
\]

(2)

The state space of the normal form is divided into two halves \( L \) and \( R \). \( \tau_L \) is the trace and \( \delta_L \) is the determinant of the Jacobian matrix \( J_L \) of the system at a fixed point in \( R_A := \{(x, y) \in R^2 : x \leq 0\} \) and close to the border and \( \tau_R \) is the trace and \( \delta_R \) is the determinant of the Jacobian matrix \( J_R \) of the system evaluated at a fixed point in \( R_B := \{(x, y) \in R^2 : x \geq 0\} \) near the border. This representation brings down the number of parameters in each side of the borderline from 4 to 2, and hence simplifies the analysis. In the present paper we will adopt this representation in investigating the occurrence of high periodic orbits in piecewise smooth maps.

Earlier research has shown that in the normal form map (2)
1. A stable period-1 orbit exists for \( \mu < 0 \) if \(- (1 + \delta_L) < \tau_L < (1 + \delta_L)\), and it exists for \( \mu > 0 \) if \(- (1 + \delta_R) < \tau_R < (1 + \delta_R)\).
2. A stable period-2 orbit exists for \( \mu > 0 \) if

\[
\begin{align*}
\tau_R &< -(1 + \delta_R), \\
\tau_R \tau_L &< (1 + \delta_R)(1 + \delta_L), \\
\tau_R \tau_L &> -(1 - \delta_R)(1 - \delta_L).
\end{align*}
\]

(3)\(\quad\)\(\quad\)\(\quad\)\(\quad\)

(4)\(\quad\)\(\quad\)

(5)

It exists for \( \mu < 0 \) if \( \tau_L \approx -(1 + \delta_L) \), and (4) and (5) are true.

3. A period-2 orbit cannot coexist with a period-1 attractor.

Therefore, in order for a multiple attractor bifurcation to occur, a HPO (high-period orbit of periodicity greater than 2) must
coexist with either a period-1 orbit, or a period-2 orbit, or another high-period orbit.

We now concentrate on the conditions of occurrence of HPOs. In the following discussion we will name particular types of HPOs depending on the partitions (L or R) in which the points fall. For example, an LLR orbit implies a period-3 orbit with two points in the left side and one point in the right. Suppose this orbit has the points \((x_{L1}, y_{L1}), (x_{L2}, y_{L2}), \) and \((x_{R1}, y_{R1})\).

In order to calculate the location of the fixed point \((x_{L1}, y_{L1})\), one has to solve for the condition \((x_{L1}, y_{L1}) \mapsto (x_{L2}, y_{L2}) \mapsto (x_{R1}, y_{R1}) \mapsto (x_{L1}, y_{L1})\). The locations of the other points are similarly found. The condition of existence of this orbit is given by \(x_{L1} < 0, x_{L2} < 0,\) and \(x_{R1} > 0\).

Using the normal form map, we obtain these conditions as

\[
\begin{align*}
(1 + \tau_R - \delta_L + \tau_L \tau_R + \delta_R \delta_L + \delta_R \tau_L - \tau_R \tau_L^2)\mu & < 0, \\
(1 + \tau_R - \delta_R + \tau_L \delta_R + \delta_R \tau_L + \tau_R \tau_L - \tau_R \tau_L^2)\mu & < 0, \\
(1 + \tau_L - \delta_L + \tau_L \tau_R + \delta_L \tau_R + \tau_L \delta_R - \tau_L \tau_R \tau_L)\mu & < 0, \\
(1 + \tau_L - \delta_L + \tau_L \tau_R + \delta_L \tau_R + \tau_L \delta_R - \tau_L \tau_R \tau_L)\mu & > 0.
\end{align*}
\]

The condition of stability of this orbit is given by eigenvalues of the composite matrix \(J_L J_L J_R\), which must be inside the unit circle. If \(\tau\) is the trace and \(\delta\) is the determinant of the above matrix, then the criteria for stability are as follows:

If \(\tau^2 < 4\delta\), i.e., if the eigenvalues are complex, then

\[
(9)
\]

\[
\delta < 1,
\]

If \(\tau^2 > 4\delta\), i.e., if the eigenvalues are real, then

\[
\begin{align*}
\tau & < (1 + \delta), \\
\tau & > -(1 + \delta).
\end{align*}
\]

For the LLR orbit these conditions are obtained by substituting

\[
\begin{align*}
\tau &= \tau_R \tau_L^2 - \tau_R \delta_L - \delta_R \tau_L - \delta_L \tau_L \\
\delta &= \delta_L^2 \delta_R
\end{align*}
\]

The orbit actually occurs in the parameter space region that satisfies the above inequalities. It is found that inequalities (8), (9), and (10) play no role in defining the contours. However, condition (10) plays an important role in eliminating a non-contiguous region where only this condition is violated.

In this manner one can obtain the condition of existence of each high-period orbit — which is the set of parameter values for which all of the above conditions (existence as well as stability) are satisfied. In this paper we refrain from presenting these conditions for various high period orbits because these are quite large. However, it should be mentioned that we have explicitly derived all these conditions and used them to obtain the regions of parameter space where each high period orbit will occur. We present these results in the next section.

There is one result that greatly simplifies this analysis. Suppose a \(n\)-periodic orbit is given by the equations:

\[
\begin{pmatrix}
x_{k+n} \\
y_{k+n}
\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_k \\
y_k \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}
\]

Then by solving for the \(n\)th periodic points we have the coefficient matrix of the state vector \((x^*_n, y^*_n)^T\) as

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 - a & -b \\ 1 - c & -d \end{pmatrix}
\]

The determinant of this matrix, which appears in the denominator of all the existence conditions, is \(1 + \alpha \delta - \beta \delta - (a + d)\). This is nothing but \((1 + \text{determinant} - \text{trace})\) of the composite matrix. By condition (10) this function will always be positive if the orbit is to be stable. Therefore, in order to check the occurrence of specific periodic orbits, one needs to look only at the numerator of the existence conditions.

Since in a continuous map single periodic orbits cannot come into existence or go out of existence, the periodic orbits always come into existence in pairs. Whenever an \(n\)-periodic orbit is found to occur in a piecewise smooth map, it can come into existence either through a saddle-node bifurcation or a border collision pair bifurcation. In a piecewise linear map, only the latter can be involved in the occurrence of a high-periodic orbit.

Therefore, whenever any particular orbit is found to occur, it must be accompanied by another unstable orbit, called the complementary orbit. This orbit has the following properties:

1. It must have the same periodicity as the observed periodic orbit.
2. Its symbol sequence differs from that of the stable periodic orbit by only one letter. However, sequences of the same letter cannot form a valid orbit. For example, for the stable LRR orbit, the complementary orbit is LRR (but not LLL).
3. Whenever the high period orbit coexists with a stable period-1 orbit, the stable manifold of the complementary orbit forms the basin boundary.

The condition of occurrence of a border collision pair bifurcation

\[
-(1 + \delta_{\text{HPO}}) < \tau_{\text{HPO}} < 1 + \delta_{\text{HPO}}
\]

and

\[
\tau_{\text{complementary}} > 1 + \delta_{\text{complementary}}
\]

is a necessary (but not sufficient) condition for the occurrence of a HPO.

We follow the classification strategy of [2], where the border collision bifurcations are described depending on the types of fixed points involved in the border collision. The eigenvalues of the normal form \(\lambda_{1,2} = \frac{1}{2} (\tau \pm \sqrt{\tau^2 - 4\delta})\) indicate that the type of fixed point changes at \(\tau\) values equal to \(-(1+\delta), -2\sqrt{\delta}, 2\sqrt{\delta}\), and \((1+\delta)\). The bifurcation behavior over the \(\tau_L\) parameter space is described by marking regions where the values of \(\tau_L\) and \(\tau_R\) lie between the above ranges.

In Fig. 1 we present two representative pictures of the region of occurrence of the LRR orbit, obtained from the existence and stability conditions given in the last section. In the earlier investigations on border collision bifurcations [2] it had been conjectured that multiple attractor bifurcation cannot occur if the eigenvalues in both the sides are real. Fig. 1 shows that this conjecture does not hold for all the regions in the parameter space where the eigenvalues are real. In Fig. 1(a) the region of occurrence of the LRR orbit is seen to intrude into the area satisfying the condition \(2\sqrt{\delta_L} < \tau_L < 1 + \delta_L\) and \(\tau_R < 1 + \delta_R\), i.e.,
where a regular attractor fixed point collides with the border and turns into a flip saddle. In the small area where that intrusion happens, a period-1 attractor bifurcates into a period-2 attractor plus a coexisting period-3 attractor as the parameter $\mu$ is varied from a negative value to a positive value.

We now turn to the other possible high-periodic orbits, whose regions of occurrence in the parameter space are worked out in a similar manner. We define “regular orbits” as those orbits having the symbols L and R occurring consecutively at least in one cyclic permutation.

We have found the following general trends in the occurrence of the HPOs.

1. For $\mu > 0$, three types of regular periodic orbits are predominantly found to occur: (a) those with one point in the right side and the rest in left side (the $L^n-1R$ orbits), (b) those with two points in left side and rest in right side (the $L^2R^{n-2}$ orbits), (c) those with two points in R and the rest in L (the $L^n-2R^2$ orbits). For $\mu < 0$, the predominant regular orbits are of $LR^{n-1}$, $L^n-2R^2$ and $L^2R^{n-2}$ types.

2. Progressively higher periodic orbits of the above types occur in monotonic progression with respect to the variation of the parameters $\tau_L$ and $\tau_R$.

3. For the regular orbits, the complementary orbit is obtained by changing one symbol in such a way that all L’s or R’s are consecutive.

To illustrate the above results, we present in Fig. 2 typical views of the parameter space showing the regions in which various periodic orbits occur.

Note that each regular orbit belong to two classes. For example, the LLR orbit is a member of the class $L^n-1R$ as well as that of class $L^2R^{n-2}$. In Fig. 2(c), the period-3 orbits are found to occur in two places (marked as LLR and RLL respectively). This implies that the region of occurrence of each regular orbit is the union of the two regions belonging to the respective classes.

We observe that each class of regular orbit occur only in a specific region of the parameter space. The gross regions of occurrence of a few classes of regular orbit are given in Table I.

It is found that for any specific combination of $\delta_L$ and $\delta_R$, regular orbits occur in monotonic progression of periodicity. For example, for any specific value of $\tau_R < -(1 + \delta_R)$ and positive values of $\mu$, if $\tau_L$ is increased, a period adding cascade of type $L^n-1R$ is observed. The cascade continues up to a specific periodicity that depends on the values of $\delta_L$, $\delta_R$, and $\tau_R$. For the same values of $\delta_L$ and $\delta_R$, the period-adding cascade progresses to a higher periodicity for a lower value of $\tau_R$. Similarly for the $L^2R^{n-2}$ orbits, if $\tau_R$ is varied, then a period adding cascade is observed which continues for a specific periodicity.

Since the progressively higher periodic orbits occur in monotonic progression in $\tau_L$ and $\tau_R$, the next question is: Do they accumulate and cease to occur beyond some values of the parameters?

To probe this question, we observe that for $L^n-1R$ orbits, the region of occurrence is bounded by the existence condition of the L point that maps to the right (which, for the LLR orbit is given by (7)), and the stability condition (11). The corner point is given by the intersection between these two curves. When we plot the location of these points for progressively higher periodic $L^n-1R$ orbits, we notice that these points lie on a regular curve that asymptotes to a line $\tau_L = k \times 2\sqrt{\delta_L}$.

Therefore, for a specific combination of $\delta_L$ and $\delta_R$, if $\tau_L$ is varied, one sees a period-adding cascade that terminates after a specific periodicity depending on the value of $\tau_R$.

What is the limiting value of $\tau_L$ beyond which this type of orbit does not occur? We find that this limiting value depends on the magnitude of $\delta_L$, and so long as $\delta_R < 1$, it is not significantly affected by the value of $\delta_R$.

From the equations for each orbit belonging to this class, we find that with the increase in the periodicity $\mu$, the curves for the existence and stability conditions move to the right. But the rate at which the existence condition moves is faster than the rate of movement of the stability condition. After a specific value of $n$ (depending on the values of $\delta_L$, $\delta_R$ and $\tau_R$), the region of intersection shrinks to zero size, and periodic orbits of higher periodicity do not occur.

For the $L^2R^{n-2}$ orbits, the regions of occurrence of each orbit is bounded by two curves. The upper bound is given by the existence condition of the left point mapping to the other left point (for the LLR orbit this is given by (6)), and the lower bound is given by the stability condition (11). Both the curves shift upwards for progressively higher periodicities, but the rate of shift is higher for the lower bound given by (11). Beyond a specific value of periodicity, higher periodicities do not occur.

When the determinants at both sides of the border are less than unity, regular orbits only of type $LR^{n-1}$, $L^n-1R$, $L^2R^{n-2}$, $L^{n-2}R^2$ are found to occur. But if the determinant in one side is greater than unity, the regular orbits of type $L^3R^{n-3}$ also occur.


but in smaller regions of the parameter space.

There are some periodic orbits that do not fall into the category of any of the “regular” orbits. The occurrence of such orbits are rare, but not impossible. Note that all possible orbits up to period 4 are regular orbits, and our numerical investigation indicates that irregular orbits of periodicity higher than 5 do not occur. In period-5 there can be two types of irregular orbits: LRLRR type, and LLRLR type. These orbits occur only for $\tau_L < 0$ and $\tau_R < 0$ (See Fig. 2).

The knowledge obtained through this investigation opens up the possibility of devising design methodologies that places the eigenvalues in practical systems such that multiple attractor bifurcations are not encountered.

### Table I

<table>
<thead>
<tr>
<th>Type of orbit</th>
<th>bounds of $\tau_L$</th>
<th>bounds of $\tau_R$</th>
<th>sign of $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^nR^{-1}$</td>
<td>$-1 &lt; \tau_L &lt; (1 + \delta_L)$</td>
<td>$-\infty &lt; \tau_R &lt; 0$</td>
<td>positive</td>
</tr>
<tr>
<td>$L^2R^{-2}$</td>
<td>$-\infty &lt; \tau_L &lt; 0$</td>
<td>$-1 &lt; \tau_R &lt; (1 + \delta_R)$</td>
<td>positive</td>
</tr>
<tr>
<td>$L^{-2}R^2$</td>
<td>$-1 &lt; \tau_L &lt; (1 + \delta_L)$</td>
<td>$-\infty &lt; \tau_R &lt; 0$</td>
<td>negative</td>
</tr>
<tr>
<td>$LR^{-1}$</td>
<td>$-\infty &lt; \tau_L &lt; 0$</td>
<td>$-1 &lt; \tau_R &lt; (1 + \delta_R)$</td>
<td>negative</td>
</tr>
<tr>
<td>Other types</td>
<td>$0 &lt; \tau_L &lt; \infty$</td>
<td>$-1 &lt; \tau_R &lt; (1 + \delta_R)$</td>
<td>positive</td>
</tr>
<tr>
<td></td>
<td>$-1 &lt; \tau_L &lt; (1 + \delta_L)$</td>
<td>$0 &lt; \tau_R &lt; \infty$</td>
<td>negative</td>
</tr>
</tbody>
</table>

Fig. 2. The regions of occurrence of high periodic orbits for (a) $\delta_L = 0.45$, $\delta_R = 0.5$ and $\mu = 0.2$, (b) $\delta_L = 0.45$, $\delta_R = 0.5$ and $\mu = -0.2$, (c) $\delta_L = 0.85$, $\delta_R = 0.9$ and $\mu = 0.2$, and (d) $\delta_L = 0.85$, $\delta_R = 0.9$ and $\mu = -0.2$.