Stochastic Resonance in Nonlinearly Coupled Langevin Equations

V.M. Gandhimathi, K. Murali, S. Rajasekar

Abstract—**We consider the nonlinearly coupled Langevin equations with the external periodic force $f\sin\omega t$ and Gaussian noise term $\eta(t)$ added to one of the two state variables of the system. First, the system with external noise term alone is studied. In the absence of forcing, the system is found to exhibit intermittent jumping motion between two wells above a particular value of noise strength D, $D_{\rm c}$. The noise-induced jumping behaviour is characterized by first-passage time and residence time distributions. Next, the influence of noise in the presence of the external periodic force is numerically studied. The system is found to exhibit stochastic resonance behaviour and is characterized using power spectrum, signal-to-noise ratio and residence time distribution. The plot of maximal Lyapunov exponent versus noise strength has shown a stochastic resonance profile. The occurrence of stochastic resonance is studied by varying the forcing frequency ω and the coupling strength δ.**

Keywords—**Stochastic resonance; Gaussian white noise; Langevin equations; Signal-to-noise ratio**

I. Introduction

THE enhancement of the output signal-to-noise ratio (SNR) in nonlinear dynamical systems via stochastic resonance phenomenon has received considerable attention in the past one decade or so [1]-[5]. Experimental and theoretical work have shown that the simplest system exhibiting stochastic resonance consists of signal and noise with a threshold-triggered device. When the signal plus noise exceeds the threshold, the system responds momentarily, then relaxes to equilibrium to await the next triggering event. Recently, the presence of weak noise is found to provide functional benefit in human performance namely balance control [6], in the individual prey capture by Juvenile Paddlefish [7], in memory recall mechanism [8] and as a means for improving signal detection [9] and information transmission [10].

In the present paper, we consider the nonlinear system

$$\begin{aligned} \dot{x} &= a_1x - b_1x^3 + \delta xy^2 + f\sin\omega t + \eta(t), \\ \dot{y} &= a_2y - b_2y^3 + \delta x^2y, \end{aligned} \qquad (1)$$

where $\eta(t)$ represents the noise term. We choose $\eta(t)$ as a Gaussian white noise with intensity D. When $\delta = 0$, the first equation in (1) is the Langevin equation which is used as a typical system to study stochastic resonance phenomenon. Equation (1) also represents overdamped version of two coupled anharmonic oscillators. In the absence of external periodic force, noise and damping terms the potential of the two coupled anharmonic oscillators is

$$V(x,y) = -\frac{a_1}{2}x^2 + \frac{b_1}{4}x^4 - \frac{a_2}{2}y^2 + \frac{b_2}{4}y^4 - \frac{\delta}{2}x^2y^2. \qquad (2)$$

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We choose a_1, a_2, b_1, b_2, $\delta > 0$. For this choice, the potential has four minima and it is a four well potential. For convenience, we designate the potential well with $x > 0$, $y > 0$ as V_{++}; $x > 0$, $y < 0$ as V_{+-}; $x < 0$, $y > 0$ as V_{-+}; $x < 0$, $y < 0$ as V_{--}. Throughout our study, we fix $a_1 = 1.0$, $a_2 = 1.1$, $b_1 = 1.0$, $b_2 = 1.0$.

II. Dynamics Of The System With Noise Only

In the absence of external periodic force eq.(1) is integrated numerically with the fourth-order Runge–Kutta method for $\delta = 0.01$ and with step size $\Delta t = 0.06$. Noise term $\eta(t)$ is implemented as $\sqrt{\Delta t}D\zeta(t)$. Here $\zeta(t)$ represents Gaussian random numbers with zero mean and unit variance and D stands for noise intensity. With an initial condition chosen in the well V_{++}, random motion confined within the single well V_{++} is realized for small values of the noise intensity D. Noise-induced jump from the well V_{++} to V_{-+} is observed for $D \geq D_{\rm c} \approx 0.03$. The numerically computed $D_{\rm c}$ values for $\delta = 0.2$, 0.35 and 0.5 are 0.05, 0.08 and 0.11 respectively. To characterize the above intermittent dynamics, we calculated the first-passage time $\tau_{\rm FP}$ from the well V_{++} to V_{-+} and the residence time $\tau_{\rm R}$ in the well V_{++}.

We have chosen the initial conditions as $(x(0), y(0)) = (1.00554, 1.05362)$ which correspond to the minimum of the potential well V_{++} for the chosen values of the parameters. We calculated the time needed to make a first transition to the well V_{-+} in the presence of the added noise term. The above calculation is repeated with same initial conditions but for 10^5 different realization of Gaussian random numbers. Then, we computed the probability distribution of first-passage time and the mean first-passage time ($\tau_{\rm MFP}$). With increase in noise intensity D, the probability distribution curve is found to move towards the origin and the magnitude of probability near the origin increased. We have studied the variation of $\tau_{\rm MFP}$ as a function of D for four fixed values of δ. In all the cases $\tau_{\rm MFP}$ decreases with increase in D from $D_{\rm c}$. From the best curve fit we found $\tau_{\rm MFP} \approx \alpha e^{\beta/(D-D_{\rm c})}$. The values of $(D_{\rm c}, \beta)$ obtained for $\delta = 0.01$, 0.2, 0.35 and 0.5 are $(0.03, 0.4)$, $(0.05, 0.5)$, $(0.08, 0.65)$ and $(0.11, 0.9)$ respectively. The value of the exponent β increases with increase in δ.

Next, we consider the residence time $\tau_{\rm R}$ defined as the duration of time the trajectory of the system resides in a well (for example, V_{++}) before switching to another well (V_{-+}). Residence time is calculated for a set of 10^5 transitions. In the intermittent region the residence time on each well depend on the noise intensity D and is randomly distributed. As D is increased, the residence time on the single well decreases. Mean residence time $\tau_{\rm MR}$ is also calculated by averaging over a set of 10^5 residence times. Power-law variation of mean residence time with $(D - D_{\rm c})$ is found.

III. Stochastic Resonance

In the previous section, we studied the effect of noise term without the external periodic force $f\sin\omega t$ in eq.(1). Now, we consider the eq.(1) in the presence of periodic force and noise term $(f \neq 0,\ D \neq 0)$. We fix $\delta = 0.01$ and $\omega = 0.05$. For these values of the parameters, in the absence of noise, the critical value of f at which cross-well periodic motion occurs first time is found to be $f_{\rm c} = 0.409$. We fix $f = 0.2$ so that in the absence of noise there is no cross-well motion. For $\omega = 0.05$, eq.(1) is integrated with the fourth-order Runge–Kutta method with step size $\Delta t = (2\pi/\omega)/N$, $N = 2000$ and the dynamics is studied as a function of the parameter D.

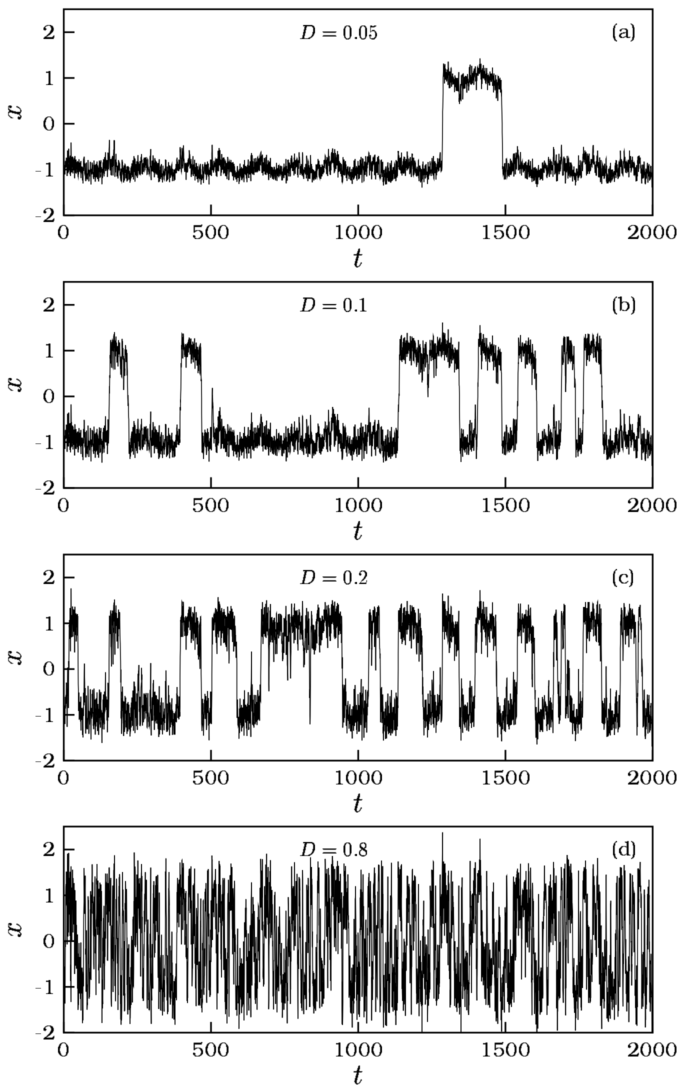

Fig. 1. x versus t for various values of noise intensity where $\delta = 0.01$, $f = 0.2$, $\omega = 0.05$, $a_1 = 1.0$, $a_2 = 1.1$, $b_1 = 1.0$, $b_2 = 1.0$.

Figure (1) shows time series plot for four values of D. For very small values of D, namely $D < D_{\rm c} = 0.02$ the motion is confined to a single well alone. At $D = 0.02$ cross-well behaviour between V_{++} and V_{-+} is initiated. The switching between the two wells is found to increase with increase in D. Intermittent hopping between the wells is clearly seen in Figs.(1a–c). The state variable x switches irregularly between noisy bands, that is between the two wells. The mechanism for the observed intermittent jumping between the noisy bands is that in the presence of noise the system initially in a well is forced by the Gaussian noise to leave the well when the trajectory is taken across the barrier. Then the system wanders irregularly in the second well and stays there for sometime and jumps back to the other well when the trajectory crosses the barrier and so on. For sufficiently large values of D, the motion is dominated by the noise. In this case, intermittent dynamics disappears and the trajectory jumps erratically between the wells. This is shown in Fig.(1d) for $D = 0.8$.

To identify the stochastic resonance behaviour, we numerically calculated power spectrum and signal-to-noise ratio for a range of values of the parameter D. Figure (2) depicts the power

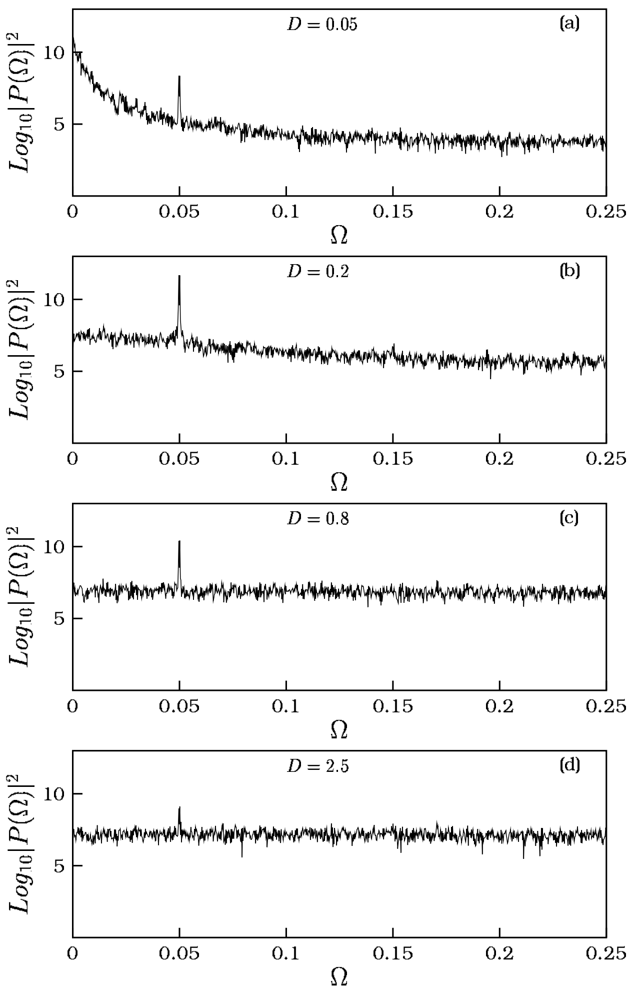

Fig. 2. Power spectral density of x-component of eq.(1) for different values of noise intensity D.

spectra for five different values of D. In all the subplots, the power spectrum shows a peak at the driving frequency $\omega = 0.05$ of the system riding on a broad noise background. The interplay between noise and periodic driving force results in a sharp increase of the signal power spectrum about the forcing frequency ω. For very low noise, the spectrum peak is low. But as we increase the noise intensity D, the height of the peak increases, attains a maximum at $D = 0.2$ and then decreases with further increase in D. This is a significant observation indicating the possibility of the occurrence of stochastic resonance.

Signal-to-noise ratio is defined as

$$SNR = 10 \log_{10}(S/N) \text{ dB}. \quad (3)$$

In eq.(3) S represents the amplitude of the signal peak and N represents the amplitude of the noise background at the driving frequency. S can be used as a measure of the response of the system to the external driving force. Equation (3) gives the value of SNR in units of decibel. Figure (3) shows the plot of SNR as a function of D. As D increases from D_{c}, the noise

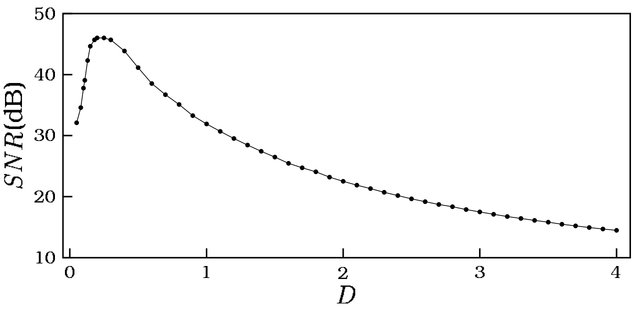

Fig. 3. Signal-to-noise ratio (SNR) as a function of input noise intensity for $\delta = 0.01$, $f = 0.2$, $\omega = 0.05$, $a_1 = 1.0$, $a_2 = 1.1$, $b_1 = 1.0$, $b_2 = 1.0$.

and signal levels increase. The noise level attains a maximum at $D = 0.1$. When D is increased further the noise level is almost flat. On the other hand, the signal level increases for D values above D_{c} and attains a maximum value at $D = 0.2$ and then begins to decrease. Though for D in the interval [0.02, 0.1] both the signal and noise levels increase, the former increases relatively at a higher rate. As a result the SNR increases with noise intensity D and peaks at $D_{\max} = 0.2$. For $D > D_{\max}$, the noise level is almost constant whereas the signal level decreases. Consequently, the SNR decreases with D for $D > D_{\max}$. For D values just above D_{c}, the time series plot shows rare switching between the wells. That is, for low noise intensity the combination of noise and external periodic force occasionally gives the system a kick sufficiently large to cross the barrier between the two wells. As D increases, at 0.2 transition between the two wells is induced for almost over every period of the driving signal. That is, nearly periodic switching with periodicity closely that of the driving force (see fig.(1c)) occurs at $D = 0.2$ which resulted in the maximum of SNR in Fig.(3). As D is further increased, switchings become frequent and irregular so that the power in the Fourier spectra is distributed widely over wide range of frequencies thereby leading to a decrease in SNR.

Probability distribution of normalized residence times is also used to characterize the stochastic resonance behaviour. This is obtained as follows. For a fixed noise intensity D, residence time τ_{R} on each well is computed for a set of 10^5 transitions. Then, normalized residence times are obtained by dividing τ_{R} by T_0, where $T_0 (= 2\pi/\omega)$ is the period of the weak periodic force $f \sin \omega t$. The distribution of normalized residence times is shown in Fig.(4) for four different values of noise intensity D with $\omega = 0.05$ and $\delta = 0.01$. The distribution shows a sequence of strong, Gaussian like peaks centered near the discrete set $\tau_{\mathrm{R}}/T_0 = n + 1/2$, $n = 0, 1, 2, \ldots$. For values of D nearly above D_{c}, τ_{R}/T_0 is distributed relatively over wide interval of time. As D increases the range of τ_{R} decreases and hence $P(\tau_{\mathrm{R}})$ of smaller τ_{R} increases. This happens upto $D = D_{\max} = 0.2$. P_1, the height of the peak at the half driving period, increases from a small value and reaches a maximum at $D = D_{\max}$. As D is further increased, P_1 decreases.

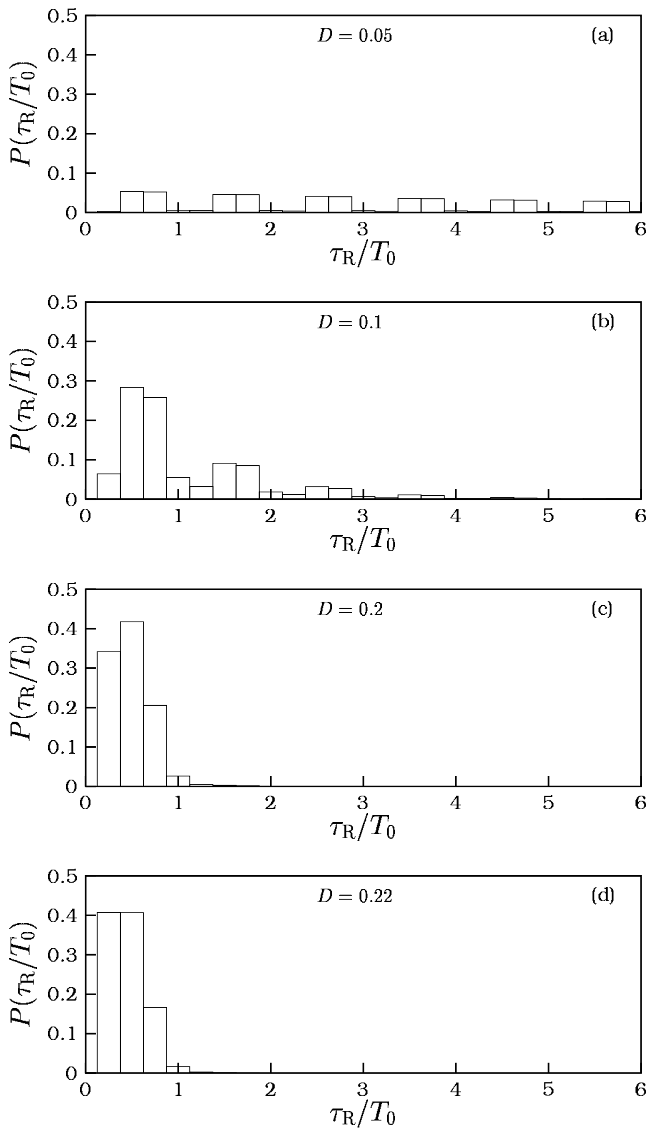

Fig. 4. Normalized residence time distribution for different values of noise intensity D.

Lyapunov exponents are calculated over 10^3 drive cycles for 10^3 different realizations of noise to obtain the average Lyapunov exponents for a fixed noise intensity D. The plot of maximal Lyapunov exponent against noise intensity D resembled the stochastic resonance profile. The maximal Lyapunov exponent is found to be negative for the range of D we considered and it increased with the noise intensity D from $D_{\rm c}$, reached a maximum value at $D_{\max}$ and then decreased.

We have calculated SNR for a range of values of D for few fixed values of ω and δ. Figure (5a) show SNR versus D for three values of ω, namely, 0.01, 0.05 and 0.1 with $\delta = 0.01$. Stochastic resonance is realized for these values of ω. For $\omega = 0.01, 0.05, 0.1$, $D_{\max}$ is found to be 0.1, 0.2, 0.32 respectively. $D_{\max}$ is found to increase with increase in ω while the maximum value of SNR is found to decrease with increase in ω.

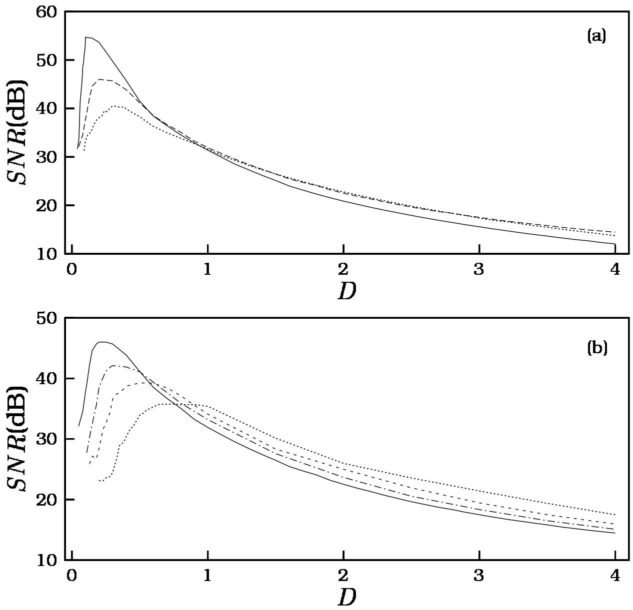

Fig. 5. (a) Signal-to-noise ratio (SNR) as a function of input noise intensity for different values of ω. The continuous, dashed and dotted curves are for $\omega = 0.01, 0.05, 0.1$ respectively. (b) Signal-to-noise ratio (SNR) as a function of input noise intensity for different values of δ, namely, 0.01 (continuous curve), 0.2 ($-\cdot-$ curve), 0.35 (dashed curve) and 0.5 (dotted curve) where $\omega = 0.05$.

Figure (5b) shows the effect of coupling strength δ on stochastic resonance. As δ increases, $D_{\rm c}$ and $D_{\max}$ also increase while the maximum value of SNR decreases. The possibility of occurrence of stochastic resonance for several fixed values of noise intensity and varying the parameters δ and f are numerically studied. With the noise intensity fixed at $D = 0.04$, f is varied from 0 to 20 for a fixed δ and δ from 0 to 1 for a fixed f. With increase in f for a fixed δ, the signal peak at the driving frequency is found to increase. For a fixed f as δ is varied from 0, the signal peak is found to decrease.

IV. CONCLUSION

In this paper we have studied numerically the effect of Gaussian noise in the two coupled Langevin eq.(1). Addition of external Gaussian noise to the unforced system shows that the system can make transitions between the wells V_{++} and V_{-+} above a critical value of noise intensity D. Power-law variation of mean residence time with $(D - D_{\rm c})$ is found. The system (1) has shown stochastic resonance behaviour and is characterized using power spectrum, signal-to-noise ratio, normalized residence time distribution and maximal Lyapunov exponent. The effect of coupling strength δ and the forcing frequency ω on the response of the system is also studied. Stochastic resonance is observed for several values of ω and δ used in the present work. The values of D_c, $D_{\max}$ and SNR at $D_{\max}$ are found to depend on the values of the control parameters ω and δ.

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