Dynamical systems model for entrainment due to coherent structures

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Abstract—**The life-time of a turbulent shear flow is inversely related to the amount of surrounding fluid it entrains. A dynamical systems model of such a flow, dominated by coherent structures, was recently proposed [1], where an axisymmetric heated jet was modelled by a pair of leap-frogging vortex rings. In the present paper, we study an equivalent but simpler flow: namely, two two-dimensional vortex pairs. The two objectives are to gain a better theoretical understanding, and to study the effect of viscous diffusion. The nonintegrability of this flow is analysed, and a proof based on splitting of seperatrices will be presented at the meeting. We perform direct simulations by vortex methods to study the effect of diffusion of the vorticity due to viscosity.**

Keywords—**Entrainment, flow topology, buoyancy**

I. INTRODUCTION

Chaotic advection in fluid flow has been of great interest, since it enables the flow to entrain surrounding fluid, the streamlines to stretch and the components to mix, even if flow velocities are low. Chaotic advection in vortical flows can be especially appealing, since even when the basic flow is periodic [2] the Lagrangian dynamics of fluid particles can be chaotic. Shear flows in a quiescent surrounding medium often impart momentum to, and carry along with themselves, quantities of surrounding (ambient) fluid. This is termed as *entrainment*, and the entrained fluid could then mix with the original fluid to modify the shear flow dynamically. An understanding of this process could be crucial in determining how pollutant is dispersed in the atmosphere, how dangerous the trailing vortex system of a landing aircraft can be to a follower aircraft, how clouds can retain water vapour as they rise through the atmosphere, to quote only a few examples. Shear flows are often dominated by the presence of coherent vortical structures. Dynamical systems models of these flows can offer simple explainations of features such as mixing and entrainment. Our purpose is therefore to study model vortical flows under the dynamical systems framework to quantify entrainment and the resulting chaos.

In a recent paper [1], it was shown that a small level of *buoyancy* (which can be achieved by volumetrically heating the flow relative to the surroundings) can modify the entrainment drastically, and on occasion even shut off entrainment completely. The predictions made in the paper were found to be in qualitative agreement with those made by direct numerical simulation [4] and experiments [5], emphasizing the validity of modelling such flows from the point of view of dynamical systems. In the present study, the objectives are twofold: (i) to prove that this system is nonintegrable, to derive conditions for the existence of fixed points and to relate this to the entrainment. (ii) to study the effects of viscous diffusion which serves to smear out the vorticity. The inviscid system is periodic in time, and consists of two leap-frogging vortex pairs (LFVP). The chaotic nature of the LFVP flow was numerically demonstrated by Pentex *et al.*[3]. The presence of hyperbolic fixed points was found responsible for the chaotic nature of the flow. The transport of fluid particles and entrainment of the ambient fluid into the flow are natural consequences of the tangling of the manifolds arising from the hyperbolic fixed points. The resulting flow dynamics is area and orientation preserving [2], these and other properties of the manifolds can be exploited to obtain the entrainment quantitativaly, and to estimate the chaos and mixing in the system.

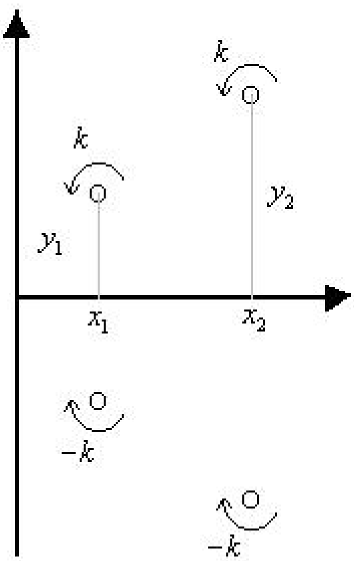

Fig. 1. Configuration of the LFVPs.

The governing equations that describe the motion of LFVP's

R. Govindarajan (corresponding author), e-mail: rama@jncasr.ac.in (see Figure 1) are Hamiltonian, and since the flow is symmetric, the Hamiltonian for the symmetry-reduced problem can be written as follows:[3]

$$H(x_i, y_i) = \frac{k^2}{2\pi} \ln\left(4y_1y_2\frac{(x_1-x_2)^2+(y_1+y_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}\right) = E. \quad (1)$$

and

$$k_i\dot{x_i} = \frac{\partial H}{\partial y_i}; \qquad k_i\dot{y_i} = -\frac{\partial H}{\partial x_i}. \quad (2)$$

The energy E of the system is a constant. The streamfunction of the flow is given by

$$\Psi(x,y,t) = -\Sigma_j \frac{k_j}{\pi} \ln r_j(t), \quad (3)$$

where

$$v_x(x,y,t) = \frac{\partial \Psi}{\partial y}, \quad \text{and} \quad v_y(x,y,t) = -\frac{\partial \Psi}{\partial x} \quad (4)$$

are the streamwise and normal components respectively. Further, a coordinate system moving along with the average velocity of the vortices is chosen so as to simplify the analysis. In the new coordinate system, two saddle points P_1 and P_2 are located in the system centreline, at $y = 0$. From P_2, an unstable manifold W_u emerges in the normal direction, while a stable manifold W_s approaches P_1 as shown in the Fig 2. A detailed review of the properties of manifolds can be found in Wiggins.[6]

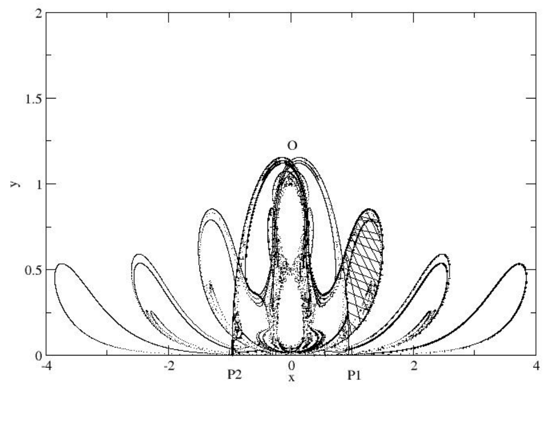

Fig. 2. The stable and unstable manifolds are plotted. If the manifolds intersect each other once, it guarantees that they shall intersect infinitely many times, which gives rise to the stretching behavior of the lobes shown in this figure. The area of each lobe is the same as all the others. [1] The vortices are denoted by dots.

By definition, a fluid particle located on a manifold stays on it througout its evolution in time. This immediately implies that the occurrence of one intersection of these manifolds ensures their repeated intersection infinitely many times. These heteroclinic intersections lead to the formation of smale horseshoes, giving rise to chaotic flow in the neighborhood. The area of the lobes (shown as the shaded region in Fig. 2) is preserved with time evolution. Further, since the orientation of each lobe with respect to the manifold remains the same, the topology of the flow is preserved, resulting in the entrainment of fluid from the external region to the internal and vice-versa. These properties, and the fact that a manifold cannot intersect itself, lead to complicated stretching as shown in Fig 2. A close look at Fig 2 also reveals the presence of KAM tori around each of the vortices. These are closed curves in the Poincare section, serving as complete barriers to flow, and therefore playing an important role in deciding entrainment and mixing. The region of interest, into and out of which entrainment is measured, is taken to be that bounded by the curves P_1O, OP_2, and P_1P_2. The Poincare section is taken here at the phase where the vortices are aligned at $x = 0$, but the results do not depend on this choice.

II. EFFECT OF BUOYANCY

The relative heating or cooling of portions of a flow results in buoyancy playing a lead role in chaotic advection. We consider the vortices to be heated or cooled relative to the surroundings, and therefore moving faster or slower, as would happen in a heated jet or cloud. An immediate effect of buoyancy on the flow topology can be seen in the change in the distance between the fixed points. If the direction of buoyancy velocity is in the same direction as the induced velocity, the fixed points come close to each other, and eventually vanish above a critical velocity, thus shutting off the entrainment. If the buoyancy velocity is opposite in sense to the induced velocity, the fixed points move further apart, and above a critical negative velocity this again leads to a reduction in the entrainment. This implies a critical buoyancy velocity at which the entrainment will attain a maximum. The buoyancy velocity (U_b) is nondimensionalized with the average streamwise velocity of the vortices in the absence of buoyancy (U_s). Fig 3 illustrates the manifolds at various buoyancy velocities. The change in the manifold structure clearly illustrates the effect of buoyancy on entrainment. Fig 4 depicts the entrainment at various pair spacings z, where z is the maximum streamwise distance during the leap-frog cycle. As predicted, one finds a maximum for every given initial pair spacing. A plot illustrating the entrainment rate is shown in Fig. 5. The maximum is almost the same for all pair spacings.

A detailed comparison with the axisymmetric case of two leap frogging vortex rings (LFVR), is being carried out. The primary difference is that in the LFVR, the self-induced velocity can be independently controlled by adjusting the ring thickness, so fixed points can vanish even without buoyancy.

III. NONINTEGRABILITY OF FLOW DUE TO LFVP'S

Bagrets *et al.* [7] have addressed the problem of nonintegrability of more than two leap frogging vortex rings using the method of separation of sepratrices (more commonly known as the Melnikov technique). In the present paper, we follow an approach similar to theirs for the proof of nonintegrability of two-dimensional flow due to the LFVP. As far as we know, the

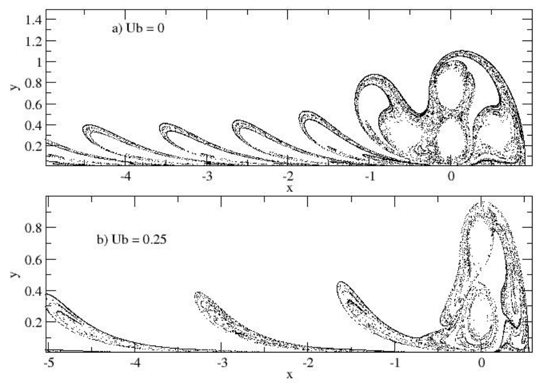

Fig. 3. The presence of buoyancy is seen to decrease the entrainment in the present case, correlated with fixed points approaching each other.

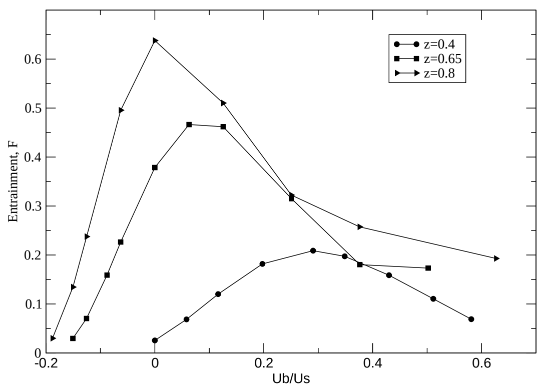

Fig. 4. Entrainment as a function of the buoyancy velocity. For each pair spacing, a maximum in the entrainment occurs at an optimum buoyancy velocity.

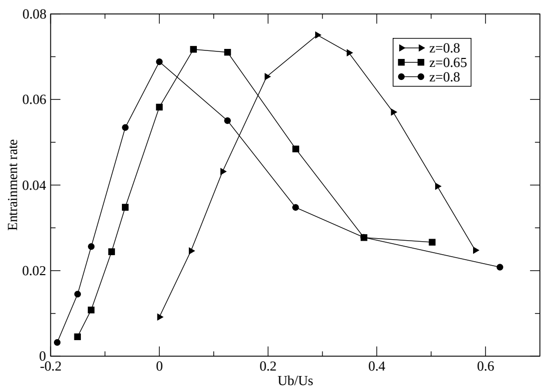

Fig. 5. The entrainment *rate* is shown here for the same parameters as in Fig. 3. The maximum in the entrainment rate is independent of pair spacing.

proof has not appeared before.

The system is considered as a perturbation from a single vortex pair, i.e., the streamwise distance x is much smaller than the normal spacing y. Accordingly, a scaling transformation for the variables in Eqs. (1)-(4) is done as follows:[6]

$$x_i = \epsilon \widetilde{x}_i, \tag{5}$$

$$y_i = 1 + \epsilon \widetilde{y}_i, \tag{6}$$

$$t = \epsilon^2 \widetilde{t}, \tag{7}$$

$$x = \epsilon \widetilde{x}, \tag{8}$$

and

$$y = \epsilon \widetilde{y} + 1. \tag{9}$$

Here, π/k is taken to be 1. Due to shortage of permitted space, we briefly describe the steps taken and state the final form directly. A complete proof will be presented at the meeting.

1. The Eqs. (1)-(4) are recast in terms of the scaled variables.
2. A coordinate transformation with the average velocity of the vortices and a rotation with the angular velocity arising from the zeroth order term of Eqs. (1) and (2) are performed.
3. Terms of $O(\epsilon)$ are shown to vanish, requiring a second order analysis of the scaled variables.

After considerable algebra, and book-keeping of order of terms, we are left with the following Hamiltonian system correct to $O(\epsilon^2)$ in the maximum streamwise distance ϵ between the vortex pairs:

$$H = H_0 + \epsilon^2 H_2, \tag{10}$$

where

$$H_0 = -\log|z-1| - \log|z+1| + \frac{1}{4}|z|^2; \tag{11}$$

$$H_2 = Re\left(\frac{Z_{12}e^{-\frac{it}{2}}}{z-1}\right) - Re\left(\frac{Z_{12}e^{-\frac{it}{2}}}{z+1}\right) + \frac{1}{4}\left\{(x^2-y^2)\cos(t) - 2xy\sin(t)\right\}. \tag{12}$$

The quantity z is redefined as $z = x + iy$, and $z_{12} = z_{12}(t) = x_{12} + iy_{12}$ is the complex representation of the second order correction of the position of the vortices. The equations governing their dynamics reduce to

$$\frac{dx_{12}}{dt} = \frac{\sin(t)}{2}x_{12} - \frac{\cos(t)}{2}y_{12} - \sin(t/2). \tag{13}$$

$$\frac{dy_{12}}{dt} = -\frac{\cos(t)}{2}x_{12} - \frac{\sin(t)}{2}y_{12} - \cos(t/2). \tag{14}$$

The Hamiltonian H_0 has hyperbolic fixed points and Melnikov's analysis may be performed on the seperatrices arising out of these fixed points. The complete analysis is similar to [7] from this point sonwards and will be presented at the meeting.

IV. Direct simulations

To study the effect of viscosity on the entrainment process we have carried out direct numerical simulations of the two-dimensional Navier-Stokes equations using a diffusion velocity discrete vortex method [8]. A vortex pair injected into the quiet ambient is simulated by a bunch of vortex blobs released from an exit nozzle. Two pairs with a known seperation between them are released and their trajectories are computed and stored. Fig 5 shows instantanious locations and sizes of two pairs just after the second one has been injected. The arrows represent the magnitude and direction of the local velocities and colors indicate the vorticity. We can now compute trajectories for a set of particles and monitor their entry, exit and residence time within a region defined in the vicinity of the vortices and moving with them. The simulations will be carried out at different Reynolds numbers, varying pair-spacing, different buoyancy velocities. Poincare sections and entrainment characteristics will be presented at the meeting.

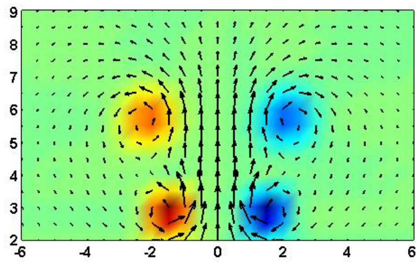

Fig. 6. Instantaneous locations of two leapfrogging vortex pairs simulated by the diffusion vortex method. The color represents the vorticity field and the arrows local velocities.

V. Summary

1. A two-dimensional dynamical systems model illustrating the effects of buoyancy on entrainment is investigated. A critical buoyancy velocity at which the entrainment attains a maximum is observed. The effect of flow topology on stretching, mixing and entrainment is demonstrated.

2. A method for the proof of nonintegrability of the motion due to LFVP is outlined. A scaling transformation is done and the first order terms are found to vanish in the final Hamiltonian. A Melnikov's analysis of the Hamiltonian including the perturbation up to second order will be presented at the meeting.

3. Numerical simulations taking into account the effects of viscosity are performed. The effect on entrainment will be presented at the meeting.

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VI. References

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