Birth of Quasiperiodicity through Border Collision in a Power Electronic Circuit

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I. INTRODUCTION

BIFURCATIONS arising from switched dynamical systems are important phenomena not only in theoretical context but also in practice. Many practical systems, such as switching power converters [1], [2], [3], chaos generators, variable structure controllers, etc. can be modeled as switched dynamical systems. The analysis of bifurcations in these systems has been found very useful in determining the possible operating regimes as well as in locating the relevant operating boundaries [3].

Switched dynamical systems are described by differential equations having discontinuous right-hand side. The evolution is described by different sets of differential equations in different compartments of the state space, and the system switches from one set to another as some conditions on the state variables are satisfied. Dynamic behaviour exhibited in such systems are of different nature than that in smooth dynamical systems. Numerous works deal with the oscillatory modes in dynamical systems with discontinuous right-hand sides [4], [5].

In this paper we investigate the novel case of the behaviour of a voltage-mode controlled buck converter with pulse width modulation of the first kind. The system switches between two sets of autonomous differential equations, and since the switching is controlled by an external periodic signal, the overall system is nonautonomous. We show that a direct transition from periodicity to quasiperiodicity is possible in this system through a border collision bifurcation. Such mechanism of the birth of a torus was earlier reported in multilevel pulse-width modulated converters. In this paper we show that the same type of transition is also possible in a much simpler and widely used power electronic circuit configuration.

To analyze the dynamical behaviour of the system, periodic solutions and their stability need to be studied. For the bifurcation analysis of the system under consideration, we use a numerical technique based on a general method proposed in [6], [7] which is applicable to switched dynamical systems. Using this method we show that the birth of torus is associated with a discontinuous jump of the eigenvalues from inside the unit circle to the outside.

II. VOLTAGE-MODE CONTROLLED BUCK CONVERTER WITH PWM-1

![PWM-1 voltage-mode controlled buck converter.](image)

We consider a voltage-mode controlled buck converter as shown in Fig. 1. It consists of a controlled switch $S$ (MOSFET), an uncontrolled switch $D$ (diode), an inductor $L$, a capacitor $C$, and a load resistance $R_L$. The switching of the MOSFET is controlled by a feedback logic known as pulse width modulation of type-1 (PWM-1). This is achieved by obtaining a control voltage $v_{con}$, as a linear combination of the output capacitor voltage $v_o$, and a reference signal $V_{ref}$ in the form

$$v_{con} = A(V_{ref} - v_o/\alpha),$$

where $A$ is the gain of the error amplifier and $\alpha$ is the factor of reduction of the output voltage $v_o$. An externally generated saw-tooth voltage $v_{ramp}$, of time period $T$ and upper and lower threshold voltages $V_U$ and $V_L$ respectively, is used to determine the switching instants. The value of the control voltage at the beginning of each ramp is compared with $v_{ramp}$. The switch is turned on at the beginning of the clock period, and is turned off when $v_{con} \mid_{at\;clock} = v_{ramp}$. The inductor current ramps up during the “ON” time, and falls during the “OFF” time. If the inductor current reaches zero value before the next clock cycle, there is a transition from continuous conduction mode (CCM) to discontinuous conduction (DCM) mode of operation.

There are three states of the system, described by three sets of differential equations, as follows.

**S0**: the equations for the “ON” state

$$\frac{d\bar{x}}{dt} = A_1\bar{x} + B_1V_{in}$$

where

$$A_1 = \begin{pmatrix} 0 & -1/L \\ 1/C & 1/RC \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1/L \\ 0 \end{pmatrix}$$
and $\vec{x} = [i_L, v_c]^T$ and $V_{in}$ is the input voltage.

$S1$: the equations for the “OFF” state

$$\frac{d\vec{x}}{dt} = A_1 \vec{x}$$

(3)

$S2$: the equations for the “discontinuous” state

$$\frac{d\vec{x}}{dt} = A_0 \vec{x}$$

(4)

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/RC \end{pmatrix}$$

The states are observed at every falling edge of the ramp signal to obtain a Poincaré map. Since the inductor current is zero in state $S2$, the system undergoes a change in dimension from 2-dimension to 1-dimension when the orbit changes from CCM to DCM.

III. BIFURCATION BEHAVIOUR OF PWM-1 BUCK CONVERTER

In this section, we will investigate the bifurcation phenomena in the voltage mode controlled buck converter. To perform the numerical analysis, the parameter values are fixed as: $L=30\mu H$, $C=22\mu F$, $R_L=10\Omega$, $A=10$, $\alpha=10$, $V_{ref}=2.5V$, $V_U=6.7V$, $V_L=1.5V$, and $T=10\mu sec$. The input voltage $V_{in}$ is varied from 60V to 30V as the bifurcation parameter. The one-parameter bifurcation diagram is generated numerically, and is shown in Fig. 2. The summary of the bifurcations obtained is as follows: For $V_{in} \geq 53.3600V$, the attractor is a period-1 orbit with small voltage ripple. The inductor current falls to zero in each clock cycle, and so the orbit is a succession of the states $S0$, $S1$, and $S2$, and the Poincaré map is effectively one-dimensional. Fig. 3(a) shows the fixed point of the Poincaré map at $V_{in}=53.3660V$. The stability of the fixed point is indicated by the fact that the system always converges to the same fixed point starting from any initial condition.

![Fig. 2. Bifurcation diagram of PWM-1 voltage controlled buck converter taking $V_{in}$ as a bifurcation parameter.](image)

By decreasing $V_{in}$, we observe the birth of a periodic orbit lying on a torus, and the transition occurs through a border collision bifurcation at $V_{in} \approx 53.3650V$. The attractor is phase-locked, and the orbit contains some clock periods in CCM and some in DCM. To verify that the attractor is really in phase-locking mode and the way by which the converter transitions from periodicity to quasiperiodicity, we sample stroboscopically (once per ramp cycle) the trajectory in the phase plane $(v, i)$. We obtain a finite number of discrete points after the steady state is reached as shown in Fig. 3(b). From physical point of view, this means that the two frequencies that govern the dynamics in the phase-locking mode are commensurable. In other words synchronization has occurred between them. On the other hand, quasi-periodic behaviour is denoted by incommensurate frequencies as shown in Fig. 3(c).

![Fig. 3. The attractors in discrete-time and in continuous-time (a) for period-1 orbit in discontinuous conduction mode when $V_{in} = 53.360V$, (b) for the mode locked period-5 orbit when $V_{in} = 49.2V$, and (c) for the quasiperiodic orbit when $V_{in} = 43.327V$.](image)

IV. THE METHOD OF BIFURCATION ANALYSIS

In this section, we explain the method of modeling the dynamical system under consideration for stability analysis of the fixed point. The system under consideration consists of 3-subsystems which can be defined as:

$$\frac{d\vec{x}}{dt} = f_i(\vec{x}, p, p_i)$$

(5)

where $f_0$, $f_1$ and $f_2$ are the functional forms of the differential equations $S0$, $S1$, and $S2$ respectively, $\vec{x} = (x_1, x_2) \in R^2$, $t \in R$, $p \in R$ are the parameters common to all the subsystems and $p_i \in R$ are the parameters specific to the subsystems $f_i$. The solutions of the sub-systems (5) can be expressed as

$$\vec{x}_i(t) = \phi_i(t, \vec{x}_i(0));$$

(6)
where \( \bar{x}_i(0) \) is the initial value for the \( i \)th subsystem. Switchings take place at every break point given by switching manifolds

\[
\Pi_i = \{ \bar{x}_i \in \mathbb{R}^2 | B_i(\bar{x}_i) = 0 \}; \quad i = 0, 1, 2
\]

(7)

where \( B_i \) are the algebraic equations defining the switching manifolds.

It is also assumed that the function changes from \( f_i \) to \( f_{i+1} \) when a solution starts from \( \Pi_i \) and reaches \( \Pi_{i+1} \) after time interval \( \tau_i \). Thus each solution of (5) evaluated at the next switching manifold can be written as

\[
\bar{x}_{i+1} = \phi_i(\tau_i, \bar{x}_i)
\]

(8)

where \( \bar{x}_i \) is the value of the state variable at the preceding switching manifold. For a period-1 orbit containing the sequence \( S0 - S1 - S2 \), the periodic solution is written as:

\[
\bar{x}_0 = \bar{x}_3 = \phi_2(\tau_2, \bar{x}_2)
\]

(9)

and the period of the limit cycle \( \tau \) is defined as

\[
\tau = \tau_0 + \tau_1 + \tau_2
\]

(10)

For the “On-Off” mode, the periodic solution is \( \bar{x}_0 = \bar{x}_2 \) and the time period \( \tau = \tau_0 + \tau_1 \). The two cases have been illustrated in Fig. 4.

![Fig. 4. Behaviour of orbit and Poincaré map.](image)

We can thus define the following local mappings (see Fig. 5)

\[
T_0 : \Pi_0 \rightarrow \Pi_1 \quad \text{given by} \quad \bar{x}_1 = \phi_0(\tau_0, \bar{x}_0)
\]

(11)

\[
T_1 : \Pi_1 \rightarrow \Pi_2 \quad \text{given by} \quad \bar{x}_2 = \phi_1(\tau_1, \bar{x}_1)
\]

(12)

\[
T_3 : \Pi_2 \rightarrow \Pi_3 \quad \text{given by} \quad \bar{x}_3 = \phi_2(\tau_2, \bar{x}_2)
\]

(13)

Here the Poincaré map is defined as the differentiable mapping described by

\[
T = T_0 \cdot T_1 \cdot T_2
\]

(14)

Note that the solution (6) is continuous, but not differentiable at the break points. The Jacobian of each local mapping can be calculated as

\[
\frac{\partial T_i}{\partial \bar{x}_i} = \frac{\partial \bar{x}_{i+1}}{\partial \bar{x}_i}, \quad \forall \ i = 0, 1, 2
\]

(15)

So Jacobian matrix of the Poincaré mapping is

\[
\frac{\partial T}{\partial \bar{x}_0} \bigg|_{t=\tau} = \frac{\partial \bar{x}_3}{\partial \bar{x}_2} \bigg|_{t=\tau_1} \cdot \frac{\partial \bar{x}_2}{\partial \bar{x}_1} \bigg|_{t=\tau_1} \cdot \frac{\partial \bar{x}_1}{\partial \bar{x}_0} \bigg|_{t=\tau_0}
\]

(16)

Here switching is modulated by the border functions of each sub-system, i.e., whenever the function \( \phi_i \) reaches the border specifically defined for the \( i \)th sub-system. In \( \Pi_0 \), the border function between the states \( S0 \) and \( S1 \) is given by

\[
B_0 : \beta_0(\bar{x}, t) = A(V_{ref} - V_C(t)/\alpha) - \frac{(V_U - V_L)T}{T} - V_L = 0;
\]

for \( t \in [0, T] \). When \( \bar{x} \) hits the border \( B_0 \), switching occurs and \( \bar{x} \) jumps to subsystem \( S1 \). Two borders exist for the subsystem \( S1 \). One is the clock signal for resetting the switch that moves the system from \( S1 \) back to \( S0 \), and other is at \( i = 0 \) which moves the system from \( S1 \) to \( S2 \) and remains there for the rest of the time interval \( \tau_2 = T - (\tau_0 + \tau_1) \) as shown in Fig. 4. These two borders can be described by

\[
B_{1a} : \beta_{1a}(\bar{x}, t) = i_L = 0
\]

(17)

\[
B_{1b} : \beta_{1b}(\bar{x}, t) = t - kT = 0
\]

(18)

While in \( S2 \), the system has only one border \( B_2 \) given by

\[
B_2 : \beta_2(\bar{x}, t) = t - kT = 0
\]

If the system hits \( B_2 \), it returns to \( S0 \).

Since in each ramp cycle the system may operate in continuous conduction mode or the discontinuous conduction mode, the model can be represented schematically, as shown in Fig. 6.

Switching conditions given by the border functions are

\[
\beta_0(\bar{x}, \tau_0) = 0, \beta_1(\bar{x}, \tau_1) = 0, \beta_2(\bar{x}, \tau_2) = 0
\]

A. Bifurcation analysis method

Stability of the period-1 solution is determined from the Jacobian of the first return map of the Poincaré mapping (14), which is given by \( \partial \bar{x}_3/\partial \bar{x}_0 \). From \( \bar{x}_3 = \phi_2(\bar{x}_2, T - \tau_0 - \tau_1) \) and using the chain rule we can write

\[
\frac{\partial \bar{x}_3}{\partial \bar{x}_0} = \frac{\partial \phi_2}{\partial \bar{x}_2} \cdot \frac{\partial \bar{x}_2}{\partial \bar{x}_1} \cdot \frac{\partial \bar{x}_1}{\partial \bar{x}_0} + \frac{\partial \phi_2}{\partial T} \left( \frac{\partial \tau_1}{\partial \bar{x}_1} \cdot \frac{\partial \bar{x}_1}{\partial \bar{x}_0} + \frac{\partial \tau_0}{\partial \bar{x}_0} \right)
\]

(19)

From (8), we get

\[
\frac{\partial \bar{x}_{i+1}}{\partial \bar{x}_i} = \frac{\partial \phi_i}{\partial \bar{x}_i} + \frac{\partial \phi_i}{\partial \tau} \cdot \frac{\partial \tau_1}{\partial \bar{x}_i}; \quad \forall \ i = 0, 1
\]

(20)
Fig. 6. Switching flow diagram of PWM-1 voltage-mode controlled buck converter.

$\bar{x}_i$ satisfies the switching condition $\beta_i(x_i, \tau_i) = 0$. Differentiating we get

$$\frac{\partial \beta_i}{\partial \bar{x}_i} + \frac{\partial \beta_i}{\partial \tau_i} \frac{\partial \tau_i}{\partial \bar{x}_i} = 0; \quad \forall \ i = 0, 1 \tag{21}$$

Solving equation (21), we get $\partial \tau_i / \partial \bar{x}_i$. Substituting it in (20) yields

$$\frac{\partial \bar{x}_{i+1}}{\partial \bar{x}_i} = \frac{\partial f_i}{\partial \bar{x}_i} \left( \frac{\partial \bar{x}_i}{\partial \bar{x}_i} \right); \quad \forall \ i = 0, 1 \tag{22}$$

Here $\partial \phi_i / \partial t$ represents $f_i(\phi_i)$ and $\partial \phi_i / \partial \bar{x}_i$ is calculated by solving the differential equation

$$\frac{d}{dt} \left( \frac{\partial \phi_i}{\partial \bar{x}_i} \right) = \frac{\partial f_i}{\partial \bar{x}_i} \left( \frac{\partial \phi_i}{\partial \bar{x}_i} \right); \quad \forall \ i = 0, 1 \tag{23}$$

with the initial condition

$$\frac{\partial \phi_i}{\partial \bar{x}_i} \bigg|_{t=0} = I_2,$$

where $I_2$ is a $(2 \times 2)$ unity matrix.

We have shown the procedure for finding out the Jacobian in case of the period-1 DCM mode of operation. For finding the Jacobian for other orbits, the specific sequence of the subsystems have to be taken into account in writing the above equations.

Using the above numerical technique, we have calculated the eigenvalues of period-1 fixed point just before and after the border collision bifurcation.

The eigenvalues of the Jacobian are 0 and -0.02635624931066 for the value $V_{in} = 53.36000V$. For $V_{in} = 53.365$, there is a discontinuous jump of the eigenvalues as shown in Fig. 7. The fixed point becomes a spiral source as its characteristics multipliers are $\lambda_{1,2} = 0.45000721649375 \pm 1.16324124472086j$, their modulus being $|\lambda_{1,2}| = 1.24725342598709 > 1$.

The results of our numerical calculations of the two-parameter diagram of the dynamical modes within the parameter plane $\{V_{in}, R_L\}$, as shown in Fig. 8. The border line shown in the plot shows the parameter values for which border collision bifurcation occurs.

Fig. 7. Discontinuous jump of the eigenvalues at the transition from period-1 to period-5 mode-locked operation.

**Fig. 8.** Domain of existence for the different oscillatory modes in the system parameter plane $V_{in} - R_L$. $N_m$ denotes the domains of existence for cycle of period m. The unmarked domains represent the region of aperiodic behaviour.

**V. Conclusion**

In this paper, we have shown the transition from periodicity to orbits lying on a torus through border collision bifurcation. We have calculated the eigenvalues of the Jacobian matrix before and after the border collision event and have shows that the eigenvalues discontinuously jump across the unit circle. We have also obtained the chart of dynamical modes over the $V_{in} - R_L$ parameter space. To our knowledge, this is the simplest practical circuit in which this new type of bifurcation has been observed.

**References**


