Effect of migration on Malaria Incidence under environmental variability-a model based approach

S. Bhattacharyya and R. R. Sarkar *

1 Introduction

Malaria is one of the most severe problems faced by the world even today. Understanding the causative factors such as migration, age, sex, social factors, environmental variability etc. as well as underlying transmission dynamics of the disease is an important epidemiological research on malaria infection and its eradication. Thus development of suitable modeling approach and methodology on the incidence of the disease and other related factors is of utmost importance. Mathematical models have been useful for estimating a proper control strategy for the disease. In recent works, there has been an emphasis on the application of control theory to epidemic models, the study of the spatial spread of diseases, the investigation of the mechanisms underlying recurrent epidemic behavior, the importance of heterogeneity in transmission, and the extension of the threshold theory to encompass more complex deterministic and stochastic models [1]-[4]. Mathematical modeling of malaria has flourished since the days of Ross, who was the first to model the dynamics of malaria transmission [5]. Macdonald expounded on Ross’s work, introduced the theory of super infection [6]. Since then, efforts have been made to model the malaria incidence using several approaches. Significant research in statistical modeling of malaria has been carried out in recent times to gauge the effect of relationship between the disease incidence and climatic factors as geographically mapping malaria risk, for example, Chattopadhyay et al. [7] proposed a regression model on P. falciparum malaria deaths in Kolkata Corporation, India, considering different environmental and social factors and compared their model with a modified version of Ross’s model. However, there is a scarcity of models which take into account the spatial heterogeneity of the environment. The few model developed indicate that, when the environment is fragmented, an increase in mobility between the patches enhances the persistence of the disease. Torres-Sorando and Rodriguez [8] have shown that the chances of establishment and the equilibrium prevalence are the same with the migration pattern and higher with the visitation pattern. Further the effect of mobility in the heterogeneous environments is not a simple matter and depends on the patterns of mobility. But so far as the literature is concerned there is no indication of how the disease incidence is affected by the migration between patches under environmental variability. In this paper, we tried to explore this issue through modelling the disease transmission process by a set of nonlinear differential equations considering the Ross’s model [5] as a basis for each patch and observing the migration between the patches under environmental variability and observed the transmission dynamics of the disease. Our results indicate that under environmental variability one-way migration helps to reduce the level of infection and thereby can be useful to control the disease prevalence under environmental noise.

2 Model and preliminaries

We consider the dynamics of malaria transmission between two patches. As a basic frame of the single patch dynamics, we follow the Ross model [5], which incorporates the interaction between the infected human hosts and infected mosquito vector population and can be expressed as a set of nonlinear differential equations. In a two patch scenario, human is more prone to migration from one patch to another patch. So, in two patch framework under human migration, model reduces to

\[
\begin{align*}
\frac{dx_i}{dt} & = \sigma_i y_i (1-x_i) - \gamma_i x_i + (D_i x_i - D_j x_j) \\
\frac{dy_i}{dt} & = a_i c_i x_i (1-y_i) - \mu_i y_i + a_i c_i D_j x_j (1-y_j)
\end{align*}
\]

where \(x_i\) and \(y_i\) are the proportions of the infected human and infected female mosquito populations at \(i\)th patch, \(a_i\) is the biting rate on human of a single mosquito, \(c_i\) is the proportion of bites by susceptible mosquitoes on infected human that produces a patent infection, \(\gamma_i\) is the individual recovery rate per human, \(\mu_i\) is the individual death rate for mosquitoes and \(\sigma_i = a_i b_i N_i / N_j\), where \(N_i\) and \(N_j\) are the (constant) sizes of the human and female mosquito populations respectively and \(b_i\) is the proportion of infected bites that produce an infection at \(i\)th patch, respectively, and \(D_i\) is the rate of migration of human from patch \(i\) to patch \(j\) (\(i,j = 1,2; i \neq j\)). We write the state variable in the matrix form

\[X := (x_1, y_1, x_2, y_2) = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}\]

We define a symmetric transformation on \(R^4\) by

\[S : \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ x_1 \\ y_2 \\ y_1 \end{pmatrix}\]

So if \(X\) is a solution of the system (1)-(2), then \(S(X)\) is also a solution. We define the following invariant subspace as

Diagonal space: \(W = \{X \in R^4 \mid x_1 = x_2, y_1 = y_2 \}\)

In the following sections, we discuss the existence of the equilibrium and the local dynamics of the above system.

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3 Equilibria and stability

For simplicity of the mathematical analysis we have considered homogenous situation for each patch and skipped the subscripts \((i,j)\) for different parameters and variables, albeit the numerical simulation has been performed for both homogenous and heterogenous situations.

3.1 Single Patch

On absence of migration, the malaria infection dynamics in a single patch is given by the following equations:

\[
\begin{align*}
\frac{dx}{dt} &= σy(1-x) - γx \\
\frac{dy}{dt} &= acx(1-y) - μy
\end{align*}
\]

(3)

(4)

In the single patch model (3)-(4) we obtain two equilibria:

(i) Trivial equilibrium \((0,0)\) and

(ii) Endemic equilibrium \(\left(\frac{acσ-μγ}{ac(σ+γ)}, \frac{acσ-μγ}{σ(ac+π)}\right)\), which exists if \(acσ - μγ > 0\), where

\[
R_0 = \frac{acσ}{μγ},
\]

(5)

is defined as the basic reproductive rate of disease transmission. It is to be noted here that,

- The equilibrium \((0,0)\) is stable if \(R_0 < 1\).
- The endemic equilibrium is stable if \(R_0 > 1\) and it is not feasible if \(R_0 < 1\), that is infection cannot persist in the system if the transmission threshold \(R_0 = 1\).

3.2 Two Patch

In two patch system, we obtain the following equilibria:

(i) Trivial equilibrium \(E_0 = \left(\begin{array}{c}
0 \\
0
\end{array}\right)\);

(ii) Nontrivial symmetric equilibrium \(E_1 = \left(\begin{array}{c}
x_1 \\
y_1
\end{array}\right)\); \(x_1 = \frac{acσ-μγ}{ac(σ+γ)}, y_1 = \frac{acσ-μγ}{σ(ac+π)}\), where

(iii) Nontrivial asymmetric equilibrium \(E_2 = \left(\begin{array}{c}
x_2 \\
y_2
\end{array}\right)\)

and \(E_2 = S(E_1)\).

The existence of positive interior equilibrium is ensured by applying fixed point theorem. A sufficient condition of existence of the equilibrium is given by

\[
0 < \frac{σ}{γ} + \frac{ac}{μ} + 2\left(\frac{1}{γ} + \frac{ac}{μ}\right) (D_1 - D_2) < \frac{1}{2}
\]

(6)

- Stability of trivial equilibrium in the diagonal space \(W\)

The variational matrix of the linearized system of (1)-(2) is given by

\[
J = \begin{pmatrix}
A_1 & B_1 \\
B_1 & A_2
\end{pmatrix}
\]

where

\[
A_1 = \begin{pmatrix}
\frac{∂F_1}{∂x} & \frac{∂F_1}{∂y} \\
\frac{∂F_1}{∂x} & \frac{∂F_1}{∂y}
\end{pmatrix}
\]

and \(B_1 = \begin{pmatrix}
\frac{∂F_2}{∂x} & \frac{∂F_2}{∂y} \\
\frac{∂F_2}{∂x} & \frac{∂F_2}{∂y}
\end{pmatrix}\)

where \(F_1 = σy_1(1-x_1) - γx_1 + (D_1x_1 - D_1x_1)\) and \(F_2 = acx_1(1-y_1) - μy_1 + acx_1 x_1(1-y_1), i,j = 1,2; i ≠ j\).

Therefore, the variational matrix for \(E_0\) is

\[
J = \begin{pmatrix}
-γ - D_1 & σ & D_2 & 0 \\
ac & -μ & acD_2 & 0 \\
D_1 & 0 & -γ - D_2 & σ \\
acD_1 & 0 & ac & -μ
\end{pmatrix}
\]

Now, the characteristic equation of the above variational matrix is

\[
α^4 + p_1α^3 + p_2α^2 + p_3α + p_4 = 0
\]

(7)

where \(p_1 = 2γ + 2μ + D_1 + D_2\)

\(p_2 = (γ + D_1)μ + (γ + D_2)μ - 2acσ - D_1D_2 + (γ + D_1 + μ)(γ + D_2 + μ)\)

\(p_3 = (μ + γ + D_1 - acσ)(μ(γ + D_2) - acσ) + D_1D_2(ac + μ)(ac - μ)\)

A close observation of the coefficients of equation (7) reveals that all the eigenvalues are of negative real part if all \(p_i > 0\). And this essentially reduces to the following conditions:

\[
μ(γ + D_1) - acσ > 0, μ(γ + D_2) - acσ > 0
\]

(8)

\[
acσ(2(γ + μ) + D_1 + D_2) < 2(γ + D_1)(γ + D_2) + μ^2(2γ + D_1 + D_2).
\]

(9)

After certain algebraic manipulation, (8) implies (9). So, condition (9) is sufficient to ensure the stability of \(E_0\). Also, it has exactly one eigenvalue with positive real part if \(p_i < 0\). It may be noted that \(p_i\) is always positive in our case. However, this implies

\[
acσ(2(γ + μ) + D_1 + D_2) > 2(γ + D_1)(γ + D_2) + μ^2(2γ + D_1 + D_2)
\]

(10)

It may be noted from relation (8) that stability of the disease free equilibrium \(E_0\) is highly dependent on the migration between patches, apart from the intrinsic parameter of single patch dynamics. Suppose \(R_{0,i} := \frac{R_0 - D_i}{D_i}, i = 1, 2\). So, stability of the trivial equilibrium \(E_0\) occurs, if \(R_{0,i} < 1, i = 1, 2\). This indicates that migration has significant role in control of the disease. We call this \(R_{0,i}\), as reproductive ratio for the two patch system. It is also seen that disease free equilibrium will be unstable if the relation (10) holds. A special case may be found, when \(D_1 = D_2\). This eventually says that \(E_0\) is unstable, if \(R_{0,i} > 1 + K\), where \(K\) is some algebraic expression of parameters and it is always positive. Another interpretation may be given of the relation in the same line. Suppose \(σ_0 := \frac{[2(γ + D_1)(γ + D_2) + μ^2(2γ + D_1 + D_2)]/[ac(2(γ + μ) + D_1 + D_2)]}{μ^2(2γ + D_1 + D_2)}, \) then \(σ > σ_0\) is exactly the relation (10). It is seen that increase in \(D_1\) and \(D_2\), increases the value of \(σ_0\) under fixed value of other related parameters. This immediately implies that increase in \(D_1\) and \(D_2\) decreases the region of instability of \(E_0\), or it decreases the growth rate of infected human, i.e., \(σ\) on a overall scenario. So, it may be concluded that migration indeed helps in diminishing the malaria incidence to some extent.
We simulate the model (11)-(12) under three different rates of migration like (25, 25) for two-way and (25, 0) for one way, etc. We have also considered and three different intensities of fluctuation, like (0.9, 0.9), (0.9, 0.8) and (0.9, 0.7), for migration rates in terms of population number, i.e., 25, 50 and 75 out of 10,000 human population in each patch and in case of both homogeneous and heterogeneous systems. It is seen, on a rough estimate, that the increase in infected population is approximately 0.25% for homogeneous case and 0.61% for heterogeneous case when migration rate increase by 0.25%. This value increases when we increase the migration rate. Surprisingly, disease incidence decreases with one way migration and it decreases very significantly (Figure 2).

Instead of usual stability analysis of the model as in deterministic case, we simulate the model numerically under different rates of migration and different strengths of environmental noise. The parameter values in two patches are obtained from literature ([4], [7]) and are shown in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Homogeneous</th>
<th>Heterogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ (l/day)</td>
<td>25</td>
<td>(25, 22)</td>
</tr>
<tr>
<td>$b$ (prob.)</td>
<td>0.95</td>
<td>(0.95, 0.8)</td>
</tr>
<tr>
<td>$c$ (prob.)</td>
<td>0.9</td>
<td>(0.9, 0.8)</td>
</tr>
<tr>
<td>$N$ (nos.)</td>
<td>20,000</td>
<td>(16,000, 15,000)</td>
</tr>
<tr>
<td>$M$ (nos.)</td>
<td>10,000</td>
<td>(10,000, 10,000)</td>
</tr>
<tr>
<td>$\gamma$ (l/day)</td>
<td>3</td>
<td>(3.0, 2.0)</td>
</tr>
<tr>
<td>$\mu$ (l/day)</td>
<td>13</td>
<td>(13, 10)</td>
</tr>
<tr>
<td>$\sigma$ (l/day)</td>
<td>47.5</td>
<td>(38, 26.4)</td>
</tr>
</tbody>
</table>


4 Stochastic Model

To observe the environmental variability in the abundance of mosquito populations, we modify the two patch system (1)-(2) incorporating a random fluctuation in the form of white noise in mosquito populations. Moreover, we reframe our basic system incorporating heterogeneity in intrinsic parameters of the system and compare the dynamics of malaria incidence (e.g. in terms of total infected human population) based on differentiability of parameter values in two patches. The modified model can be written in the form

$$dx_i = [a_i c, x_i (1 - x_i) - \gamma_i x_i + (D_i x_j - D_i x_i)] dt$$  \hspace{0.5cm} (11)

$$dy_i = [a_i c, x_i (1 - y_i) - \mu_i y_i + a_i c D_i x_i (1 - y_i)] dt + \omega_i d\xi_i,$$ \hspace{0.5cm} (12)

where $\omega_i$ are real constants defining as intensities of stochasticity and $\xi_i$ is a 1-dimensional white noise process having scalar Wiener process components with increments $\Delta \xi_i^j = \xi_i(t+\Delta t) - \xi_i(t)$, which are independent Gaussian random variables $N(0, \Delta t)$-distributed $(i = 1, 2)$ [9] and are independent to each patch.

4.1 Simulation scheme

We simulate the model (11)-(12) under three different rates of migration like $D_i = 25, 50, 75$ for both two-way as well as one-way migration in human population between patches, that is, (25, 25) for two-way and (25, 0) for one way, etc. We have also considered and three different intensities of fluctuation, like $\omega_i = 0.3, 0.6, 0.9$ under these several migration rates. Effect of two types of migration (for example, two-way and one-way) under different intensity of environmental noise has been articulated and compared for both homogeneous patches and heterogeneous patches. We have considered the average and standard deviations of last 5,000 out of total 10,000 realisation of 5 simulations of the system in each case and plotted the total infected human population for two patches to compare the effect of respective mechanism in each.

5 Results and Discussion

Dynamics of the malaria incidence shows intricate features in response to different migration rates. It may be interesting to note that total infected population in two patches under two-way migrations increases with increase in migration rate, even in presence of noise (Figure 1). We have chosen three different migration rates in terms of population number, i.e., 25, 50 and 75 out of 10,000 human population in each patch and in case of both homogeneous and heterogeneous systems. It is seen, on a rough estimate, that the increase in infected population is approximately 0.25% for homogeneous case and 0.61% for heterogeneous case when migration rate increase by 0.25%. This value increases when we increase the migration rate. Surprisingly, disease incidence decrease with one way migration and it decreases very significantly (Figure 2).

We have chosen the same rate of migration from patch 1 to...
patch 2. It should be noted that population density and other intrinsic parameters rate are indeed less in Patch 2 than that of patch 1. However, it is seen that disease incidence decreases 15% in case homogeneous and 17% in case of heterogeneous patches, when there is no environmental noise in the system. The reduction is more in presence of noise. A typical time series exhibiting the population of infected human and mosquito in two different homogeneous patches under environmental noise is shown in Figure 3. Both in homogeneous and heterogeneous case, it decreases by 18% on same migration rate. So, the migration may help in prevention or control the disease in this case, if we consider one way migration from higher incidence patch to lower one, instead of two way migration. However, increase in noise intensity increases the malaria incidence, both for two-way migration or one-way migration (Figure 4).

Although the average of total infected population do not increases much, but the variances are getting higher with increase in the intensity of noise. We have computed the coefficient of variation (CV) for different noise. It is seen that in case of both heterogeneous and homogeneous parametric set up and in both two-way and one-way migration, intensity of noise increases the CV significantly. Another interesting point to be noted that this CV arising due to noise decreases in case of one-way migration between heterogeneous patches, when we increases the migration rate. It can be shown that increase in migration rate in one-way framework decreases the CV, when patches are heterogeneous, albeit it increases in case of homogeneous patches. In a very realistic scenario, most unlikely that different patches will exhibit same parameter values. So, we may draw the conclusion at this point that one-way migration might help in diminishing the disease incidence, and also the variance in infected population due to noise decreases with increase in migration rate. So migration may act as control measures for prevention of the disease on a spatio-temporal scale.

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References