Dynamics of Real Verhulst Networks

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0.1 Well known scalar Verhulst System

The real Verhulst equation, also known as logistic equation, was coined to model population growth of a region by Verhulst [6], [7] . The discrete-time real Verhulst equation is:

$$x[n+1] = ax[n]\left(1 - \frac{x[n]}{K}\right) \quad (1)$$

where a is the rate of growth of population and K is the carrying capacity of the region.
At low population ($x[n] << K$), the rate of growth is exponential. As the population becomes comparable to carrying capacity, its exponential growth is hindered. The dynamics of this system is dependant on the rate a. The values of a at which the population converges to equilibrium is $1 < a < 3$. At $a > 3$ and $a < 3.56995$, bifurcation is observed and at $a > 3.56995$, chaos is observed.
This is reffered to as a scalar Verhulst system in this article since it only describes the non-linear population dynamics of a single region.
A recent development is the Grey Verhulst Model, which enables us to use training data from the real world to form a Verhulst model to predict the future course of various saturated S-form sequences like Network Security Situations [2] and the number of students taking entrance examinations [3]. This is even extended to neural networks where Grey Verhulst Models have been used to complement back propagation algorithms in time-series forecasting [8].

0.2 Main Innovation: Verhulst Network

This paper does not involve the use of Grey Verhulst models for prediction of various phenomenon with real data. Instead, we return to the original concept of a scalar Verhulst system as described by the discrete logistic equation [1] [6] [7] and we extend it over multiple regions, each with its own rate and carrying capacity. A Real Verhulst Network (RVN) is a network of such regions connected by weights. Each node in the network represents a region. Each weight represents the percentage of total population migrating from one region to another. The state of a node at the instance 'n' represents the population of that region in a discrete time duration (1 year, 10 years, etc.). Any RVN with N nodes can be characterised by the following equation:

$$x_i[n+1] = a_i \sum_{j=1}^{N} w_{ij} x_j[n] \left(1 - \frac{\sum_{k=1}^{N} w_{ik} x_k[n]}{K_i}\right) \quad (2)$$

where $i \in \{1, 2, .., N\}$.
Here, a_i is the rate of population growth for region i, K_i is the carrying capacity for region i, $x_i[n]$ is the state of node i at time n and w_{ij} is the fraction of population in region j which is immigrating to region i.

0.3 Assumptions

1) It is in discrete time.

2) The fraction of population moving from one region to another is always non-negative and does not change from the initial value assigned at $n = 0$.

3) The updation of population (or state updation) at all nodes is done in parallel.

4) If the population post-immigration exceeds carrying capacity, then population at the node becomes zero. In equation terms, if $\sum_{k=1}^{N} w_{ik} x_k[n] > K_i$, then $x_i[n+1] = 0$.

0.4 Main Contributions

1) The first novel contribution of this article is the transformation of any RVN to its base form, which makes it easier to analyse properties of any general RVN. A RVN is in its base form if it satisfies the condition that the sum of entries along any row of the weight matrix is 1, i.e. $\sum_{j=1}^{N} w_{ij} = 1 \; \forall \; i \in \{1, 2, .., N\}$.

2) The second novel contribution of this paper is the analysis of the dynamics of a RVN which is in base form, the rate of every node equals a and the carrying capacity of every node equals K. Interesting properties like homogeniety arise for this type of RVN.

3) The third novel contribution of this article is the analysis of dynamics of another type of RVN which satisfies the conditions $\sum_{j=1}^{N} w_{ij} a_i = E$ and $K_i = K \; \forall \; i \in \{1, 2, .., N\}$.

4) The fourth novel contribution of the article is observing

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patterns displayed by all RVNs in base form and arriving at suitable explanations along with real world implications. This is done via coding in C++ and subsequent simulation.

0.5 Mathematical Derivations and Theoretical Proofs

1) The non-zero equilibrium state vector for a RVN in base form satisfying conditions $a_i = a$ and $K_i = K \;\forall\; i \in \{1, 2, .., N\}$ is $\left[K\left(1-\frac{1}{a}\right), K\left(1-\frac{1}{a}\right), ..., K\left(1-\frac{1}{a}\right)\right]^T$.

2) The conditions for local convergence to equilibrium for a RVN in base form satisfying conditions $a_i = a$ and $K_i = K \;\forall\, i \in \{1, 2, .., N\}$ are $1 < a < 3$.

3) Proof by *strong mathematical induction* that the condition for local convergence to equilibrium is sufficient for any valid initial state vector ($0 < x_i[0] < K$) to converge to equilibrium for a RVN in base form satisfying conditions $a_i = a$ and $K_i = K \;\forall\; i \in \{1, 2, .., N\}$. This is done in a similar manner as done in [1] but by using a bounded function $\psi_r[n]$ to take care of the weights.

4) The equilibrium state vector for a RVN which satisfies the conditions $\sum_{j=1}^{N} w_{ij} a_i = E$ and $K_i = K \;\forall\; i \in \{1, 2, .., N\}$ is $\left[\frac{Ka_1}{E}\left(1-\frac{1}{E}\right), \frac{Ka_2}{E}\left(1-\frac{1}{E}\right), ..., \frac{Ka_N}{E}\left(1-\frac{1}{E}\right)\right]^T$.

5) The condition for convergence to equilibrium for any valid input for a RVN which satisfies the conditions $\sum_{j=1}^{N} w_{ij} a_i = E$ and $K_i = K \;\forall\; i \in \{1, 2, .., N\}$ is $1 < E < 3$.

0.6 Experimental Observations

1) The bifurcation occurs simultaneously at all N-nodes in most cases.

2) If a_j is increased, then the bifurcation patterns occur for smaller values of $a_i \;\forall\; i, j \in \{1, 2, .., N\}$. The inference is that increasing the rate at any node increases the net rate of the entire system.

3) If the weight if w_{ij} is increased and w_{ik} is decreased, the value of $a_l \;\forall\; i, j, k, l \in \{1, 2, .., N\}$ at which bifurcation occurs is *non-increasing* if $a_j > a_k$ and *non-decreasing* if $a_j < a_k$. It is important to note that this pattern is not observed universally for all cases. It is most prominent if the carrying capacities at all nodes are equal and less prominent if carrying capacities at nodes are unequal. The inference is that this operation increases the influence of node j and decreases the influence of node k, which means that effective rate of the RVN is pushed towards a_j and away from a_k.

4) We define the ratio of carrying capacities $r_{ij} = K_i/K_j$. It is observed that if r_{ij} is kept constant, then the equilibrium values of the nodes are proportional to the scale by which the carrying capacities of all nodes have been increased or decreased.

5) It is observed that if the ratio between the carrying capacities of few nodes are very large or very small, then some of those nodes may be redundant, i.e. the state of these nodes are zero while other states are non-zero at stable equilibrium or cycles of length 2^k. In real world terms, the node only acts as a death trap to populations immigrating there.

6) Semi-redundant nodes are also observed. These nodes are non-zero whenever the RVN converged to equilibrium but after bifurcation, they tend to become zero periodically within the cycle of length 2^k. In real world terms, the node acts as a death trap occasionally while sustains some population at other instances.

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