Classification of Quasi-Periodic Bifurcations of 2- and Higher-Dimensional Tori Using Lyapunov Bundles and Its Demonstration Via a Practical Electronic Circuit

T. Endo *

In continuous-time dynamical systems, a periodic orbit becomes a fixed point on a certain Poincaré section. The eigenvalues of the Jacobian matrix at this fixed point determine the local stability and type of bifurcation of the periodic orbit. Analogously, a 2-torus quasi-periodic orbit becomes an invariant closed curve (ICC) on a Poincaré section. The Lyapunov exponents corresponding to the eigenvalues of a fixed point, determine the stability of an ICC. In particular, we focus our attention to the Lyapunov exponent with the smallest nonzero absolute value which we call the Dominant Lyapunov Exponent (DLE). A local bifurcation manifests as a crossing or touch of the DLE locus with zero. However, the type of bifurcation cannot be determined from the DLE. This is because calculation of DLE uses absolute value of the expansion rate. To overcome this problem, we define the Dominant Lyapunov Bundle (DLB), which corresponds to the dominant eigenvectors of a fixed point. We prove that the DLB of an ICC (a 1-torus in a map) can be classified into four types: A^+ (annulus and orientation preserving), A^- (annulus and orientation reversing), M (Möbius band), and F (focus). The DLB of a 2-torus in a flow can be classified into three types: A^{2+}, M^2 (equivalently, $A^- \times M$ and $M \times A^-$), and F^2. For the 1-torus in a map, we conjecture that type A^+ and A^- DLBs correspond to a saddle-node and period-doubling bifurcations, respectively, whereas a type M DLB denotes a double-covering bifurcation, and type F relates to a Neimark-Sacker bifurcation. Similarly, for the 2-torus in a flow, we conjecture that type A^{2+} DLBs correspond to saddle-node bifurcations, type M^2 DLBs to double-covering bifurcations, and type F^2 DLBs to the Neimark-Sacker bifurcations [1].

Next, we demonstrate an example of double covering bifurcation for 2-torus in flow from an actual electronic circuit experiment and the associated SPICE simulation of a phase-locked loop, which looks quite unusual for most of researchers [2]. Specifically, we observe both double-covering and period-doubling bifurcations of a discrete map on two Poincaré sections, which are realized simply by changing the sample timing from one baseband sinusoidal signal to the other, confirming the double-covering bifurcation of the original 2-torus flow.

At last, we extend our results for higher-dimensional tori [3]. We propose a method for analyzing higher-dimensional tori, which uses both one-dimensional tori in sections (ST1) and zero-dimensional tori in sections (ST0). The bifurcation of ST1 can be classified into five classes: saddle-node, period doubling, component doubling, double-covering, and Neimark-Sacker bifurcations. The bifurcations of ST0 can be classified into four classes: saddle-node, period doubling, component doubling and Neimark-Sacker bifurcations. We present examples of all of these bifurcations.

References

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*T. Endo is with the Department of Electronics and Bioinformatics, Meiji University, Kawasaki 214-8571 Japan, email: endo@meiji.ac.jp