Speed fluctuations in a limit cycle

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The Morris-Lecar Model [1] and Plant Model [2] are examples of dynamical systems which are used to understand the phenomenon of bursting in neurons [3]. These models exhibit alternating phases of quiescence and bursting—typical of neuronal dynamics.

We have developed a simple non-linear model which can also capture the essence of such dynamics:

$$\dot{x} = I - y - x^2 \quad (1)$$

$$\dot{y} = y + (x^2 - 4)(x - 6) \quad (2)$$

where I is the system parameter. The dynamical characteristics of our system are that there exist three fixed points—a saddle, an unstable focus, and a stable node—for parameter values below a bifurcation value. At this value, a saddle-node bifurcation occurs and the saddle-node pair disappears.

For values of the parameter greater than the bifurcation value, a stable limit cycle is born and this limit cycle surrounds the unstable focus. We specify a point on the limit cycle by the angle θ (measured anti-clockwise) between the x-axis and the state located on the limit cycle. The plot of the speed of rotation versus position on the limit cycle (Fig.1) revealed wild oscillations in speed. What is responsible for the fluctuation in speed?

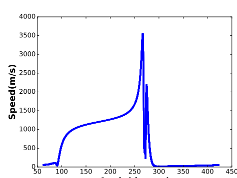

Figure 1: Variation of speed with angle.

Our conjecture is that the character of the vector field in a neighbourhood of the point where the saddle-node pair merged and disappeared, continues to influence the limit cycle. We call it a “ghost fixed point” which is assumed to exist for parameter values greater than the bifurcation value.

We have shown that such oscillations in speed occur in most non-linear systems having a stable limit cycle. To understand the origin of speed oscillations, we have divided the limit cycle into two parts: the left part lying in the neighbourhood of the unstable focus and the right part lying in the neighbourhood of the ghost fixed point. We found that the oscillations in the first part was due to the mathematical properties of the real fixed point, some other special points in this region and the structure of the dynamical equations of the system. The oscillations in the second part was found to be connected to the existence of the ghost fixed point in that region.

We found that the dynamics near the ghost fixed point can be understood by considering the quadratic approximation to the vector field in a small rectangular window containing the ghost fixed point. We assumed such a quadratic approximation in x as the functional form of the flow and considered a rectangular window of width 2ϵ. The breadth was chosen large enough to accommodate a special set of trajectories which simulated the right half of the limit cycle. The speed variation on each trajectory was characterized by the local maxima and local minima present and the sequence had a structure as a whole. This structure showed the property of transition.

We investigated similar systems of quadratic nature in x which had ghost fixed points and repeated the above procedure for them and obtained the structures. These structures were different but they also showed the property of transition. In order to understand these transitions, we derived the condition for the speed function $(v_x^2 + v_y^2)$ to have a local maxima or local minima. This condition described a set of curves in the x-y space. We found that the transitions are related to the intersections of these curves with the special set of trajectories in the window. By applying this idea to just one trajectory, we show that

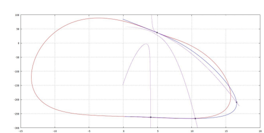

Figure 2: Approximation picture

the oscillations in speed in the right part of the limit cycle are connected with the ghost fixed point 2. The same idea applied to the whole limit cycle provides an exact mathematical explanation of the number and location of the oscillations in speed for our system.

References

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