Invariant Density of Piecewise Smooth Maps in Presence of Noise

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Discrete time piecewise smooth (PWS) maps are frequently used for representation of many physical and engineering systems [2, 3]. Such maps in n dimension are given by more than one functional form defined on non-overlapping compartments of the phase space, separated by $(n-1)$ dimensional subspaces called 'borders' where the derivative of the map is undefined. Normally we consider a piecewise linear approximation of the map in the neighbourhood of the border in order to investigate the beviour of these maps. This technique is commonly used to investigate border collision bifurcations [1, 4].

The physical and engineering systems represented by such maps always exist in noisy environment, which calls for the study of the dynamical properties of such maps in presence of noise.

In this paper we consider a one-dimensional PWS map with two compartments defined by two smooth functions separated by a single point, the 'border':

$$x_{n+1} = \begin{cases} k_1 + k_2 x_n & : x \leq x_b \\ k_3 + k_4 x_n & : x \geq x_b \end{cases} \tag{1}$$

where k_1, k_2, k_3, k_4 are constants and x_b is the border.

Now we consider an additive Gaussian noise term with the above map to take into account the effect of the noisy environment. Then our system (1) becomes

$$x_{n+1} = \left\{ \begin{array}{ll} k_1 + k_2 x_n & : x \leq x_b \\ k_3 + k_4 x_n & : x \geq x_b \end{array} \right\} + \varepsilon \xi_{n+1}, \tag{2}$$

where the random variables $\xi_n \sim N(0,1)$ and ξ_n are independent for all n. ε is the amplitude of the noise term with $0 \leq \varepsilon < 1$.

In this paper we consider that there exists an attracting fixed point of (1) in any one side of the border x_b. In absence of noise, the density function is a delta function. In presence of noise, the density function changes, which is the focus of our investigation.

Depending upon the noise amplitude and the distance of the fixed point from the border, there arise two possibilities regarding the positions of the iterates of the map (2).

1. After a finite number of iterations all the iterates remain confined on a particular side (where the attracting fixed point exist) of the border.

2. Iterates are distributed on both side of the border.

Under the assumption of the first possibility it has been shown that the density of (2) becomes a Gaussian centered about the attracting fixed point, and the functional form has been reported in [5]. In this paper we investigate the other case.

We show that the density function in this case is composed of two Gaussian functions, and obtain the functional form of each. The result closely matches the density function obtained from numerical experiment (see Fig 1)

We also consider the piecewise nonlinear maps arising in the context of impacting systems[3], whose functional form is

$$x_{n+1} = \left\{ \begin{array}{ll} k_1 + k_2 x_n & : x \leq x_b \\ k_3 + k_4 x_n^{\alpha} & : x \geq x_b \end{array} \right\} + \varepsilon \xi_{n+1}, \tag{3}$$

Where α is a real number such that $k_3 + k_4 x_n^{\alpha}$ is continuously differentiable for $x_n \geq x_b$. We derive a mathematical expression for the invariant density of (3). We also validated our theoretically obtained density functions by comparing it with numerical results, obtained by performing a long term numerical simulation.

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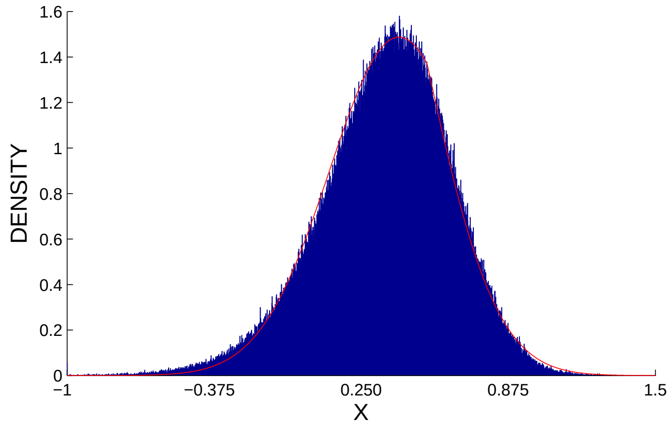

Figure 1: Invariant Density

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