State estimation using gradient descent method

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State estimation and forecast of physical systems is a topic of intense research [1]. In this study we use gradient descent method for accurate estimation of state from some set of observations of nonlinear systems like discrete Ikeda map and Lorenz system. Associated with a full set of observations there exists a set of true states of the model which are indistinguishable from each other that is known as indistinguishable states [2, 3]. Here by full set of observations we mean that all components of the model is measurable which is an ideal case. Let us consider the following discrete time dynamical systems

$$x_{t+1} = f(x_t) + w_t, \tag{1}$$

where, $x_t \in R^d$ is the d dimensional state variable at time t and $f : R^d \mapsto R^d$ provides the dynamics of the systems, and w_t are independent random numbers and referred as the *dynamical noise*. For *deterministic* systems $w_t = 0$ and the system is *stochastic* if $w_t \neq 0$. We will consider the systems is deterministic so $w_t = 0$.

Let us consider that the observation s_t of the state variable x_t at time instant t is obtained using the following rule

$$s_t = h(x_t) + \nu_t, \tag{2}$$

where, $h : R^d \mapsto R^k$ is the observational operator which maps from d dimensional model space to k dimensional observational space ($k \leq d$) and $\nu_t \in R^k$ is the *measurement noise* with density $\rho(\nu)$. Here, we will consider that the measurement errors are sampled from Gaussian distribution with mean zero and variance σ^2.

Let, $S \equiv \{s_1, s_2, \ldots, s_n\} \in R^{nd}$ represents the sequence of observations for a sequence of states $X = \{x_t\}_{t=1}^{n}$ of the dynamical systems in Eq (1). Our aim is to find a trajectory that shadows the observations, i.e. the trajectory remain close and consistent with the observation. To do this we choose the cost function which is the *indeterminism* relative to the function f

$$L(x) = \frac{1}{2} \sum_{t=1}^{n-1} ||x_{t+1} - f(x_t)||^2. \tag{3}$$

In Eq. (3), $L(x) = 0$ when x is a trajectory of the dynamical systems Eq (1). When the observation $S \equiv \{s_1, s_2, \ldots, s_n\} \in R^{nd}$ is not a trajectory, then gradient descent algorithm starting at $x(0) = S$ will follow the gradient of L in R^{nd} in the steepest descent path to a minimum value $L(x) = 0$, in equation this is equivalent of solving the following ordinary differential equation

$$\frac{dx}{d\tau} = -\nabla L(x(\tau)), \tag{4}$$

where, τ is the descent time. We solve Eq. (4) with the initial condition $x(0) = S$, and in the limit $\tau \longrightarrow \infty$, $x(\tau)$ will be the trajectory of the dynamical system.

For numerical studies Let us consider the two dimensional Ikeda map

$$u_{n+1} = 1 + 9/10(u_n \cos\theta_n - v_n \sin\theta_n) \tag{5}$$

$$v_{n+1} = 9/10(u_n \sin\theta_n + v_n \cos\theta_n) \tag{6}$$

where, $\theta_n = 2/5 - 6/(1 + u_n^2 + v_n^2)$.

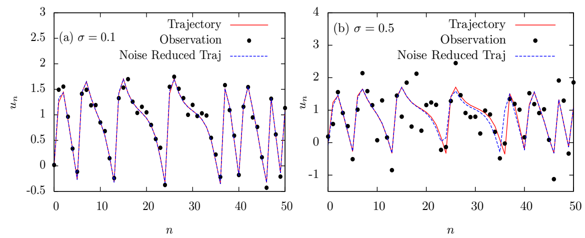

Figure 1: The true trajectory (solid red line), noisy observation (black dots), and the noise reduced trajectory (blue dashed line) are plotted as a function of iteration for Ikeda map with chaotic dynamics. (a) $\sigma = 0.1$, (b) $\sigma = 0.5$.

In Fig. (1)(a) & (b) we plot the true trajectory (solid red line), the noisy observations (black dots), and the noise reduced trajectory (blue dashed line) for two different noise realizations. In Fig.(1)(a) the observations are contaminated with measurement noise sampled from Gaussian distribution with mean zero and standard deviation $\sigma = 0.1$, and in Fig. (1)(b) the measurement noise are sampled from Gaussian distribution with mean zero and standard deviation $\sigma = 0.5$. From Fig. (1)(a) & (b), we can see that the noise reduced trajectory can estimate the true trajectory within good accuracy even with larger noise level.

References

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