Unfolding quasiperiodic trajectories

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A trajectory $x_n := F^n(x_0)$, $n = 0, 1, 2, \ldots$ is said to be quasiperiodic if the trajectory lies on and is dense in some d-dimensional torus. In such systems, there is a choice of coordinates on the torus $\mathbb{T}^d$ for which F has the form $F(\theta) = \theta + \rho \bmod 1$, i.e, the dynamics is that of a rotation. The vector ρ, called the *rotation vector* of F is the key characteristic of any quasiperiodic trajectory, and is the only prerequisite to completely determine the topological and measure theoretic properties of the dynamics. In this paper, we develop a general method for using x_n to determine ρ.

There is an extensive literature on determining the rotation number, however, even in the case $d = 1$ there has been no general method for computing ρ given only the trajectory x_n, though there is an extensive literature dealing with special cases. Here we present our *Embedding Continuation Method* for computing the components of ρ from a trajectory. It is based on the Takens Embedding Theorem and the Birkhoff Ergodic Theorem.

The Birkhoff Ergodic Theorem asserts under "typical" conditions, time averages computed along a trajectory converge to the space average. Rotation numbers can be analytically expressed as the space average of an angular measurement $\Delta\theta$ and hence is always computed using the Birkhoff ergodic theorem. We will first demonstrate that for sufficiently smooth systems, an innovative and small modification of numerical Birkhoff averages significantly speeds the convergence rate for quasiperiodic trajectories – by a factor of 10^{25} for 30-digit precision arithmetic, making it a useful computational tool for autonomous dynamical systems.

However, the variable $\Delta\theta$ is highly non-trivial to be defined, and if the embedding is complex and high-dimensional, defining $\Delta\theta$ continuously and consitently is a complex task, as an angle between two iterates can be given by multiple values. To acheive this, we introduce the Embedding Continuation Method, where we look for continuous *lift* of the angle difference between successive iterates which respects the rule that nearby points in the original torus correspond to nearby angle difference measurements.

There is however a caveat; the coordinates of ρ obviously depend on the choice of coordinates of $\mathbb{T}^d$. We explore the various sets of possible rotation numbers that are "equivalent" in the sense that they represent the dynamics upto conjugation.Ther ideas will be illustrated with examples in dimensions $d = 1$ and 2.