Random Reverse Cyclic Matrices

Shashi C. L. Srivastava,* Sudhir R. Jain† and Arul Lakshminarayan ‡

Random matrices having additional structures other than demanded by various symmetries often present interesting testbeds to study the effects of constraints on various properties like spectral distributions, spacing distributions etc[1]. Recently there has been a renewed interest in understanding how the constraint of constant column/row sum affects the density distributions and other statistical properties of eigenvalues and eigenfuntions [3, 4]. In this work, we first review the spectral distributions obtained for a class of symmetric matrices with additional structure called reverse cyclic matrices and then present some new results related with the exact distribution of maximum and minimum eigenvalues.

Random reverse cyclic matrices, that are real-symmetric, is defined by choosing elements of first row independently and then subsequent rows will be obtained by a left shift on previous row. This matrix is closely related to cyclic or what is also called circulant matrices by a permutation matrix. This fact becomes immediately clear from the definition of the matrix where again only first row elements are chosen independently and subsequent rows are obtained by a *right* shift on previous row. Using this relation and the fact that Fourier matrices diagonalize the circulant matrices, it can be easily shown that eigenvalues of H are equal in magnitude and with +ve sign for eigenvalues in upper half plane, and -ve sign for those in lower complex half plane.

For an ensemble of reverse cyclic matrices, drawn from a Gaussian distribution, $P(H) \sim \exp(-A\mathrm{Tr}H^{\dagger}H)$. For such cases the joint probability distribution function (jpdf) of eigenvalues is obtained as,

$$P(E_1, E_2, \ldots, E_{k+1}, \theta_1, \ldots, \theta_{2k}) = \left(\frac{A}{\pi}\right)^{(2k+1)/2} \times \quad (1)$$
$$|E_2| \ldots |E_{k+1}| e^{-A(E_1^2 + 2\sum_{i=2}^{k+1} E_i^2)}.$$

Spacing distribution of eigenvalues is obtained using jpdf in Eq. 1 which match nicely with all the numerical results. Details can be found in [1].

Distribution of extreme eigenvalues

The eigenvalues of a cyclic matrix is given by

$$E_l = \sum_{p=1}^{N} a_p \exp \frac{2\pi i}{N}(p-1)(l-1); \quad (2)$$

where $l = 1, \ldots, N$ these eigenvalues can equivalently be thought of being the discrete Fourier transform of first row of the matrix. As a_ps are random numbers, E_l in Eq. 2 can be interpreted as position of l^{th} Random walker after N steps in a 2-dimensional space where different directions possible for walker to jump along is given by roots of unity and length of the jump in that particular direction is given by a_p. The modulus of E_l will then give the distance travelled by l^{th} Random walker from origin after N-steps . It is clear that in a given realization each walker can be characterized by the combination of the directions chosen and not the length of step which is same for each Random walker. Hence it would be interesting to ask which walker travels maximum distance after N-step or how the records of distance travelled by Random walkers behave? Hence distance travelled by different Random walkers can be taken as time series.

As we have shown that (positive) eigenvalues of reverse cyclic matrices are precisely the same (modulus of E_l). Hence, $|E_l|$ can be taken as time-series *i.e.* $(|E_1|, |E_2|, \ldots, |E_{(N+1)/2}|)$ (for odd dimensional matrices)[1] with index i signifying the eigenvalue corresponding to i^{th} discrete Fourier basis and record statistics for such time series can be studied. The maximum of this time-series will also give the maximum eigenvalue of reverse-cyclic matrix. As reverse cyclic matrix is a symmetric matrix, one would expect that edge distribution of eigenvalues may be given by Tracy-Widom. Indeed, it has been shown in [5] that the largest eigenvalue for these class of matrices are distributed in accordance with Gumbel distribution. Using the formulation presented in [6], we are presenting record statistics of non-trivial eigenvalues of RRCM. With this, it has been shown that not only the largest eigenvalue but any record, R, once properly shifted and re-scaled follows Gumbel distribution as in Eq. 3.

$$Q(R, t) = \left(1 - e^{-R^2}\right)^t \quad (3)$$
$$\sim \exp(-\exp(-x)) \text{ (Gumbel), with} \quad (4)$$
$$x \approx 2\sqrt{\log t}(R - \sqrt{\log t}) \quad (5)$$

We have also shown that the smallest positive non-trivial eigenvalue is distributed as an exponential which is equivalent in our Random walker description to say that least distance travelled by a Random walker is exponentially distributed.

*Shashi C. L. Srivastava is with Variable Energy Cyclotron Centre, HBNI, Kolkata-700, 064, email: shashi@vecc.gov.in

†Sudhir R. Jain is with Bhabha Atomic Research Centre, HBNI, Mumbai-400 085

‡Arul Lakshminarayan is with Indian Institute of Technology Madras, Chennai-600 036

[1] the index is restricted to $(N+1)/2$ as rest of the eigenvalues will be complex conjugate of eigenvalues E_2 to $E_{(N+1)/2}$

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