Leibniz algebroid axiomatics and their role in mechanical systems

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Lie algebroids [1] a generalisation of Lie algebra have been increasingly playing an important role in Physics. For example they have been useful in formulating quantum field theories that are more general than Yang-Mills[2] and also mechanical systems more general than that of Lagrange and Hamilton [3, 4]. Recall the definition of a Lie algebroid

Definition 1 : A Lie algebroid structure on the vector bundle $E \to M$ is an R-linear bracket $[[,]]$ on the space $\Gamma(E)$ of the global cross section of the vector bundle and a bundle morphism $\varrho : \Gamma(E) \to \Gamma(TM)$ also called the anchor map, such that

1. $[[s_1, s_2]] = -[[s_2, s_3]]$.
2. $[[s_1, fs_2]] = f[[s_1, s_2]] + \varrho(s_1)(f)s_2$.
3. $\varrho[[s_1, s_2]] = [\varrho(s_1), \varrho(s_2)]$.
4. $[[s_1, [[s_2, s_3]]\,]] - [[\,[[s_1, s_2]], s_3]] - [[s_2, [[s_1, s_3]]\,]] = 0$.

Recently it has been noticed that relaxing some of the above properties can lead to "different mechanics". Namely If we were to drop the Jacobi identity then these structure go by different names in the literature like skew-symmetric algebroid structure, almost lie algebroid structure and almost lie structure. These structure now can be used to describe non-holonomic systems. Further if one were to drop the skew-symmetric condition such structures have been called Leibniz algebroids [5, 6], and they have been found to be useful in describing dissipative systems. A remark is needed here, the aforementioned Leibniz algebroids are different from the Leibniz algebroid as defined by Loday. We are interested in understanding the role of Loday/Leibniz algebroids in mechanical systems.

As a first result in our work on Leinbiz algebroids we show that the above mentioned conditions in the definition of a Lie algebroid can be reduced to conditions 2 and 4. This reduction of axioms leads to the question what mecahnics will they give rise to.

Let $E \to M$ be a smooth vector bundle with a bilinear product on $\Gamma(E)$ satisfying the Jacobi identity. Assuming only the existence of an anchor map $\mathfrak{a}$ we show that $\mathfrak{a}([X, Y]) = [\mathfrak{a}X, \mathfrak{a}Y]_c$. This gives the redundancy of the homomorphism condition in the definition of Leibniz algebroid (in particular if it arises from a Nambu-Poisson manifold); an aspect not addressed in the literature. We apply our result to the brackets of Hagiwara [7], Ibañez et. al [8]; we settle an old query of Uchino on redundancy for Courant bracket [9].

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